• $|\sigma_\tau(f,\xi) - S(\mathcal{P}(L_\tau))| = \left|\sum_{i=1}^{n} 2\pi f(\xi_i)\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i - \pi\sum_{i=1}^{n}[f(x_{i-1})+f(x_i)]\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i\right|$

$= \pi\left|\sum_{i=1}^{n}\Big(\underbrace{[f(\xi_i)-f(x_{i-1})]}_{(2)\Rightarrow<\varepsilon} + \underbrace{[f(\xi_i)-f(x_i)]}_{(2)\Rightarrow<\varepsilon}\Big)\right|\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$

$\le \pi\sum_{i=1}^{n}\Big(|f(\xi_i)-f(x_{i-1})| + |f(\xi_i)-f(x_i)|\Big)\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$ ★

• $f(x) \in$ непр. в/у $[a,b] \to f(x)$ е равном. непр $\Rightarrow$

$\varepsilon>0,\ \exists\ \delta' = \delta'(\varepsilon) > 0 : \forall x', x'' \in [a,b]:$

$|x'-x''| < \delta' \Rightarrow |f(x')-f(x'')| < \varepsilon$

$$\left.\begin{matrix}|x_{i-1}-\xi_i| \le \Delta x_i \le \delta_\tau < \delta \\ |x_i-\xi_i| \le \Delta x_i \le \delta_\tau < \delta\end{matrix}\right. \Rightarrow \left\{\begin{matrix}|f(x_i)-f(\xi_i)| < \varepsilon \\ |f(x_{i-1})-f(\xi_i)| < \varepsilon\end{matrix}\right\} (2)$$

★ $< \pi\sum_{i=1}^{n} 2\varepsilon\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$ ★★

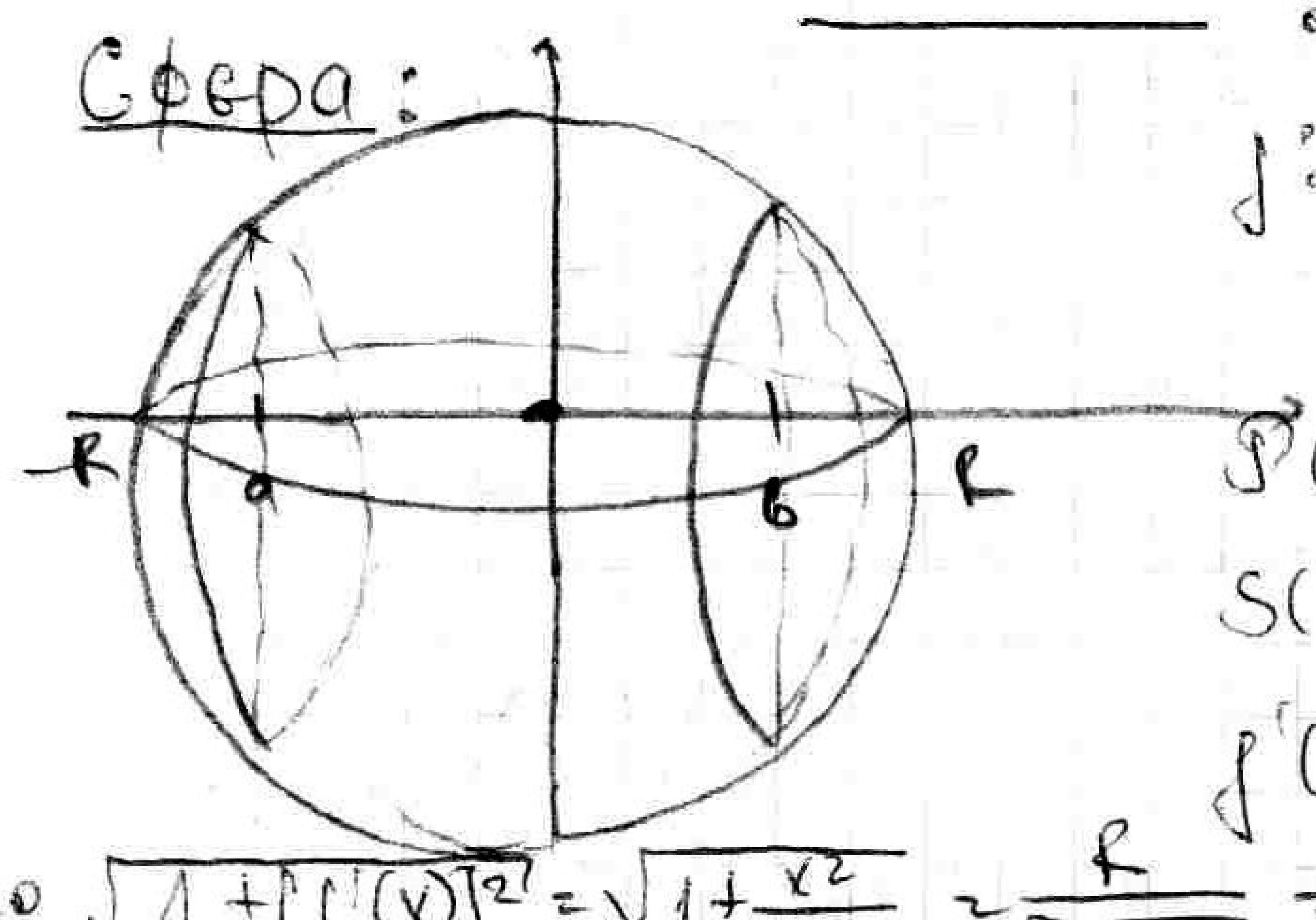
• $\sqrt{1+[f'(x)]^2} \in$ непр. в/у $[a,b] \to$ е огр в $[a,b]$, т.е. $\exists\ C>0:$

$\forall x\in[a,b] : \sqrt{1+[f'(x)]^2} \le C$ (3)

$\overset{(3)}{\Rightarrow}$ ★★ $< 2\pi\varepsilon C\sum_{i=1}^{n}\Delta x_i = \underbrace{2\pi(b-a)\cdot C}_{const}\cdot\varepsilon < [1+2\pi(b-a)C]\,\varepsilon$

$S(\mathcal{P}(f)) = \lim S(\mathcal{P}(L_\tau)) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

Сфера:



$f: x^2+y^2 = R^2 \Rightarrow y = f(x) = \sqrt{R^2-x^2},$

$x\in[a,b] \subset [-R,R]$

$\mathcal{P}(f)$ - сфера с рад. R и ц-р $(0,0)$

$S(\mathcal{P}(f)) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

$f'(x) = \left(\sqrt{R^2-x^2}\right)' = -\frac{2x}{2\sqrt{R^2-x^2}} = -\frac{x}{\sqrt{R^2-x^2}}$

• $\sqrt{1+[f'(x)]^2} = \sqrt{1+\frac{x^2}{R^2-x^2}} = \frac{R}{\sqrt{R^2-x^2}} \Rightarrow$

$f(x)\sqrt{1+[f'(x)]^2} = \cancel{\sqrt{R^2-x^2}}\cdot\frac{R}{\cancel{\sqrt{R^2-x^2}}} = R$

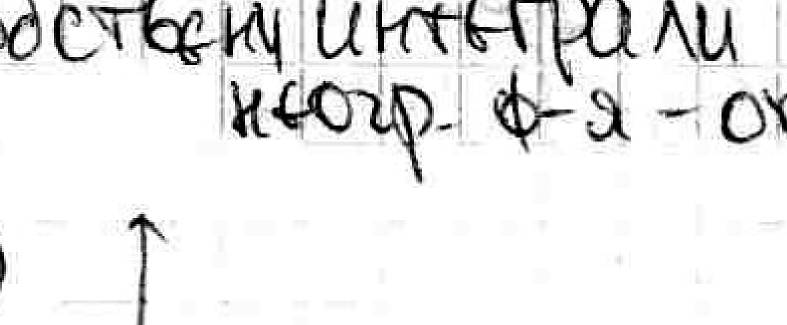
$\Rightarrow S(\mathcal{P}(f)) = 2\pi\int_a^b R\,dx = 2\pi R(b-a)$ - лице сферичен пояс (защр.)

• Ако $[a,b] = [-R,R]$ получ. лицето на сфера

$$\boxed{S_{сф} = 4\pi R^2}$$

(11) Несобствени интеграли в/у безкраен интервал и от неогр. ф-я - опр., св-ва.

Примери: 1)



$f(x) = \frac{1}{1+x^2},\ x\in[0,+\infty)$

$D = \{(x,y): x\ge 0,\ 0\le y\le \frac{1}{1+x^2}\}$ - неогр. м-во защр.

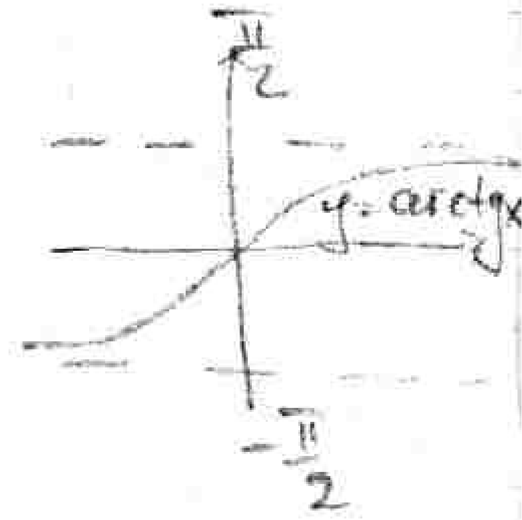
$S(D) = ?$

$0 < \xi \to$ произв. число

$D_\xi=\{(x,y): 0\le x\le \xi,\ 0\le y\le \frac{1}{1+x^2}\}$

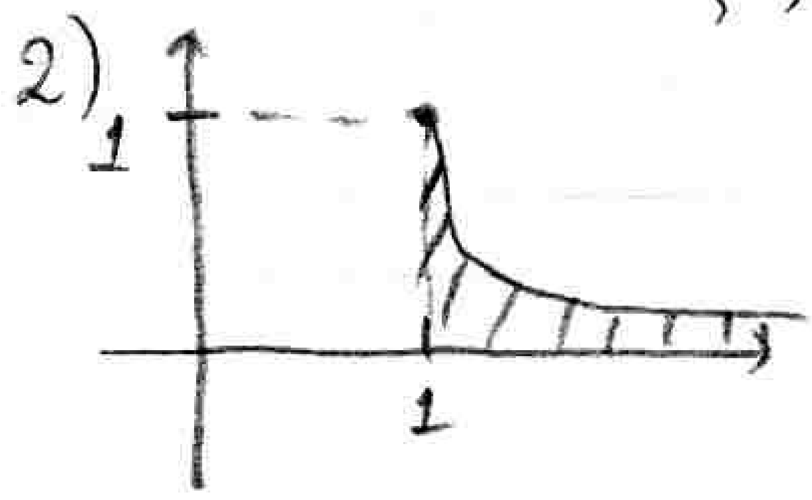
$S(D_\xi)=S(\xi)$

$S(\xi)=\int_0^\xi \frac{1}{1+x^2}\,dx = arctg\,x\Big|_0^\xi = arctg\,\xi - arctg\,0 = arctg\,\xi$



$\lim S(D_\xi)=\lim_{\xi\to+\infty} S(\xi)=\lim_{\xi\to+\infty} arctg\,\xi = \boxed{\frac{\pi}{2}=S(D)}$

2)



$f(x)=\frac{1}{x}$, $[1;+\infty)$

$D=\{(x,y): x\ge 1,\ y\in[0;\frac{1}{x}]\}$

$1\le\xi$ - произв.

$D_\xi=\{(x,y): 1\le x\le\xi,\ 0\le y\le \frac{1}{x}\}$

$\Rightarrow \exists\ S(\xi)=S(D_\xi)=\int_0^\xi \frac{1}{x}\,dx = \lim_{\xi\to+\infty}\ln x\Big|_1^\xi = \ln\xi-\ln 1=\ln\xi$

$\lim_{\xi\to+\infty} S(D_\xi)=\lim_{\xi\to+\infty} S(\xi)=\lim_{\xi\to+\infty}\ln\xi=+\infty$

$\Rightarrow \nexists$ граница

$\Rightarrow$ лицето на защр. част не може да се намери

Деф| Нека $f(x)$ - деф в/у $[a,+\infty)$ и такова, че:

$\forall \xi\ge a$, $f(x)\in$ инт. в/у $[a;\xi]$

• Ако $\exists \lim_{\xi\to+\infty}\int_a^\xi f(x)\,dx$, то казваме, че $f(x)$ е инт. в несобствен смисъл в/у $[a,b]$,

$\lim_{\xi\to+\infty}\int_a^\xi f(x)\,dx=\int_a^{+\infty}f(x)\,dx$ - несобствен интеграл (сходящ се) от $f(x)$ в/у $[a;+\infty)$

Примери: 1) $f(x)=\frac{1}{1+x^2}$ е инт. в н. см. в/у $[a,b]$

$\int_a^{+\infty}\frac{1}{1+x^2}\,dx=\frac{\pi}{2}$

2) $f(x)=\frac{1}{x}$, $[1;+\infty)$ - $f(x)$ не е инт. в н. см. в/у $[1;+\infty)$

Забележка: $f(x)$ деф в/у $[a;+\infty)$ и инт. в/у $[a;\xi]$ $\forall a\le\xi$,

$\int_a^{+\infty} f(x)\,dx$ $\quad\int_a^{+\infty}\frac{1}{x}\,dx$

Деф| 1) $f(x)$ - деф в/у $(-\infty;a]$ и инт. в/у $[\eta,a]$ $\forall\eta\le a$

• Ако $\exists \lim_{\eta\to-\infty}\int_\eta^a = \int_{-\infty}^a f(x)\,dx$ ↳ нес. инт. в/у $(-\infty;a]$

↳ инт. в н. см. в инт. $[\eta,a]$

2) $f(x)$ - деф в/у $(-\infty,+\infty)$ и инт. $\forall[\eta,\xi]$:

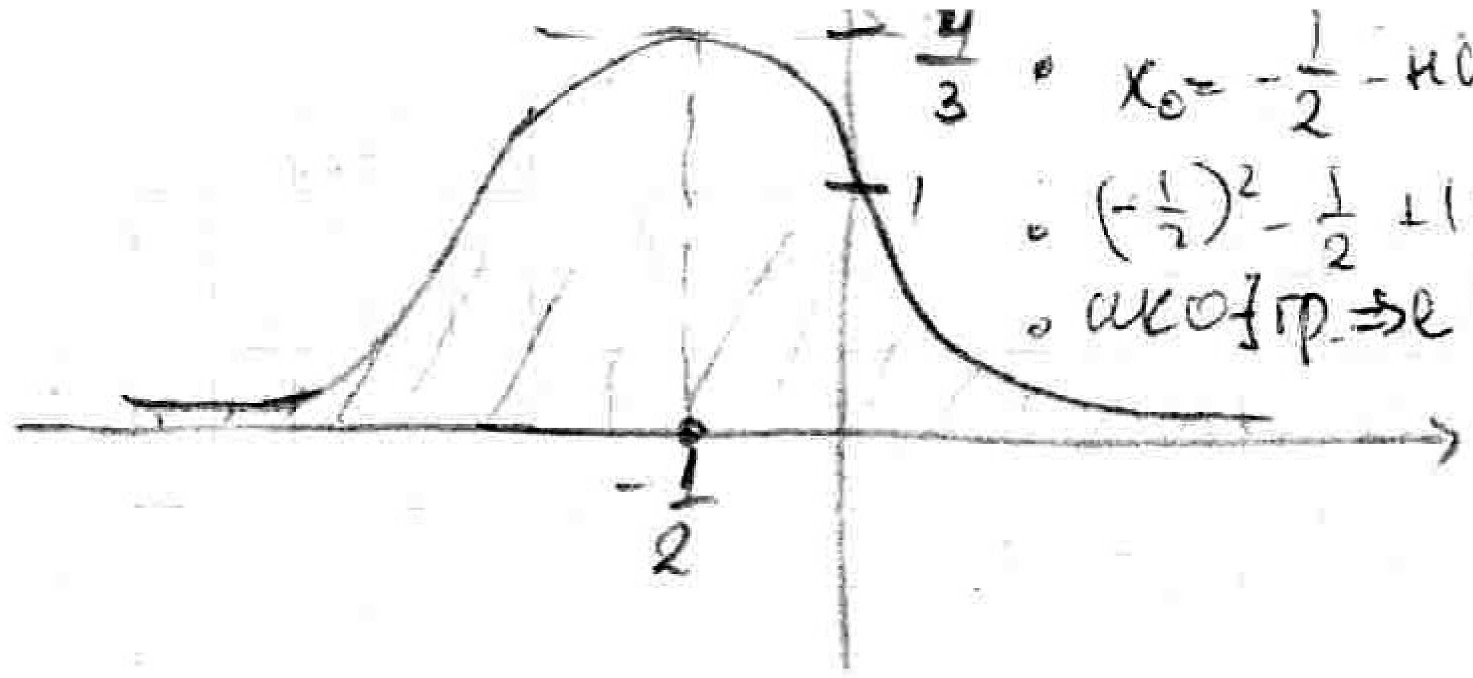
$\forall(\eta\le\xi)\Rightarrow \int_{-\infty}^{+\infty}f(x)\,dx=\int_{-\infty}^a f(x)\,dx+\int_c^{+\infty}f(x)\,dx$, $c\in\mathbb{R}$

Примери: 1) $\int_{-\infty}^0 xe^{-x^2}\,dx=\lim_{\eta\to-\infty}\int_\eta^0 xe^{-x^2}\,dx=$

$=\lim_{\eta\to-\infty}\left[-\frac{1}{2}\int_\eta^0 e^{-x^2}\,d(-x^2)\right]=\lim_{\eta\to-\infty}\left(-\frac{1}{2}\right)e^{-x^2}\Big|_\eta^0 = \left(\frac{1}{e^{\eta^2}}\to 0\right)$

$=\lim_{\eta\to-\infty}\left[-\frac{1}{2}+\frac{1}{2}e^{-\eta^2}\right]=-\frac{1}{2}+\frac{1}{2}\cdot 0=-\frac{1}{2}\Rightarrow$ е сходящ със ст-ст $-\frac{1}{2}$

2) $\int_{-\infty}^{+\infty} \frac{1}{x^2+x+1}dx =$



• $x_0 = -\frac{1}{2}$ - най-ниш

• $(-\frac{1}{2})^2 - \frac{1}{2} + 1 = \frac{4}{3}$

• ако ∃ пр. ⇒ е сх

$= \lim_{\substack{\xi \to +\infty \\ \eta \to -\infty}} \int_\eta^\xi \frac{1}{x^2+x+1}dx =$

$= \lim_{\substack{\xi \to +\infty \\ \eta \to -\infty}} \int_\eta^\xi \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} d(x+\frac{1}{2}) =$

$= \lim_{\substack{\xi \to +\infty \\ \eta \to -\infty}} \frac{4}{3} \int_\eta^\xi \frac{1}{\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]^2+1} d\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right] = \lim_{\substack{\xi \to +\infty \\ \eta \to -\infty}} \frac{2}{3} arctg\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]\Big|_\eta^\xi =$

$= \frac{2}{\sqrt{3}} \lim_{\substack{\xi \to +\infty \\ \eta \to -\infty}} \left[arctg\frac{2}{\sqrt{3}}(\xi+\frac{1}{2}) - arctg\frac{2}{\sqrt{3}}(\eta+\frac{1}{2})\right] =$

$= \frac{2}{3}\left[\lim_{\xi \to +\infty} tg \underbrace{\frac{2}{\sqrt{3}}(\xi+\frac{1}{2})}_{\to +\infty} - \lim_{\eta \to -\infty} arctg \underbrace{\frac{2}{\sqrt{3}}(\eta+\frac{1}{2})}_{\to -\infty}\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - (-\frac{\pi}{2})\right] = \frac{2\pi}{\sqrt{3}}$

⇒ инт. е сх. със ст-ст $\frac{2\pi}{\sqrt{3}}$ -> лице на защр. част. -> м/у ф-ята и Ox

3) $I(\alpha) = \int_1^{+\infty} \frac{1}{x^\alpha}dx = \begin{cases} \frac{1}{\alpha-1}, \text{ ако } \alpha > 1, \text{ сходящ} \\ \text{разходящ, ако } 0 < \alpha \le 1 \end{cases}$

$\alpha = 1$: $\int_1^{+\infty} \frac{1}{x}dx$ -> разходящ

⇒ $\alpha \ne 1$: $\int_1^{+\infty} \frac{1}{x^\alpha}dx = \lim_{\xi \to +\infty} \int_1^\xi \frac{1}{x^\alpha}dx = \lim_{\xi \to +\infty} \int_1^\xi x^{-\alpha}dx = \lim_{\xi \to +\infty} \int_1^\xi \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_1^\xi$

$= \frac{1}{1-\alpha}\left[\lim_{\xi \to +\infty} \xi^{1-\alpha} - 1\right] = \frac{1}{1-\alpha}\left[\lim_{\xi \to +\infty} \frac{1}{\xi^{\alpha-1}} - 1\right]$

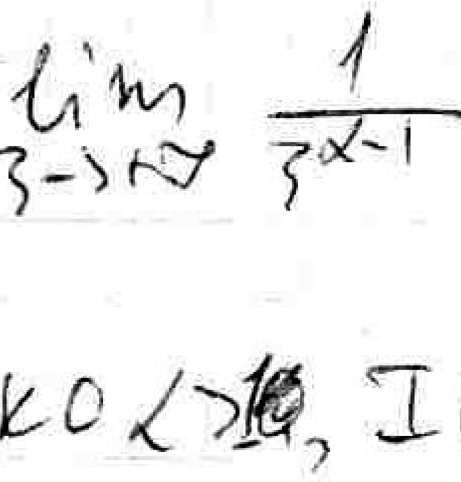
$\overset{\alpha > 1}{=} \frac{1}{1-\alpha}(0-1) = \frac{1}{\alpha-1}$

$\overset{\alpha < 1}{=} \left[\lim_{\xi \to +\infty} \xi^{1-\alpha} - 1\right] = \infty$ ⇒ Ако $\alpha > 1$, $I(\alpha)$ е сх

Ако $\alpha \in (0,1)$, то $I(\alpha)$ е разх

4) $f(x) = \frac{1}{\sqrt{1-x}}$, $x \in [0,1)$

$D = \{(x,y): 0 \le x < 1, 0 \le y \le \frac{1}{\sqrt{1-x}}\}$



$S(D) = ?$ лице на безкр. част.

- $0 \le \xi < 1$

$D_\xi = \{(x,y): 0 \le x \le \xi, 0 \le y \le \frac{1}{\sqrt{1-x}}\}$

- $S(D_\xi) = \int_0^\xi \frac{1}{\sqrt{1-x}}dx = \int_0^\xi (1-x)^{-\frac{1}{2}} d(1-x) = -\frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^\xi = -2\sqrt{1-\xi} + 2$

- $\lim_{\xi \to 1-0} S(D_\xi) = \lim_{\xi \to 1-0} \int_0^\xi \frac{1}{\sqrt{1-x}}dx = \lim_{\xi \to 1-0}[-2\sqrt{1-\xi}+2] = 2 = S(D)$

⇒ $\exists S(D) = 2$

5) $f(x) = \frac{1}{1-x}$, $x \in [0,1)$

$D = \{(x,y): 0 \le x \le 1, 0 \le y \le \frac{1}{1-x}\}$

$D_\xi = \{(x,y): 0 \le x \le \xi, 0 \le y \le \frac{1}{1-x}\}$

$S(D_\xi) = \int_0^\xi \frac{1}{1-x}dx = -\int_0^\xi \frac{1}{1-x} \cdot d(1-x) = -\ln(1-x)\Big|_0^\xi = -\ln(1-\xi) + \ln 1 = -\ln(1-\xi)$

$\lim_{\xi\to 1-0} S(D_\xi) = \lim_{\xi\to 1-0} [-\ln(1-\xi)] = +\infty$ => не можем да намерим лице на D

Деф| Нека $f(x)$ е деф. в/у $[a,b)$, $f(x)$ - инт за $\forall: a \le \xi < b$ и неогр. в/у $[\xi, b)$. Тогава, ако $\exists \lim \int_a^\xi f(x)dx$, то казв., че $f(x)$ е инт. в несобств. смисъл в/у $[a,b]$, а $\lim_{\xi\to b-0} \int_a^\xi f(x)dx = \int_a^b f(x)dx$ -> несобств. инт. от $f(x)$ в/у $[a,b]$

- с особ. в долната гр.

- $\int_0^1 \frac{1}{\sqrt{1-x}}dx$ - несобств. инт. , $\frac{1}{1-x}$ - не е инт. в несобств. смисъл

Забележка| $\int_a^b f(x)dx$: 1) $f(x)$ - деф. $[a,b]$ и $\forall\ \eta \in [a,b]: f(x)$ е инт. $[\eta, b]$ и неогр. в/у $[a,\eta]$ -> $\lim_{\eta\to a+0} \int_\eta^b f(x)dx = \int_a^b f(x)dx$ -> горната гр.

2) Ако $f(x)$ е деф в (a,b) и $\forall\ a < \eta \le \xi < b$: $f(x)$ е инт. $[\eta,\xi]$ и неогр. в/у $[a,\eta]$ и $[\xi,b)$ ->

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\quad c\in(a,b)$$

е особ. в горна и долна граница

- $\int_a^{\infty} f(x)\,dx$ - несобств.

$\int_a^b f(x)\,dx$ - тр. да пров. дали е несобств.

Примери:

1) $\int_0^1 \frac{1}{x^\alpha}dx = \begin{cases} \text{разх}, & \alpha \ge 1 \\ \text{сх}, & 0<\alpha<1 \end{cases}$ $\int_1^{+\infty} \frac{1}{x^\alpha}dx = \begin{cases} \text{сх}, & \alpha > 1 \\ \text{разх}, & 0<\alpha\le 1 \end{cases}$

$\lim_{\eta\to 0+0} \frac{1}{x^\alpha}dx = \lim_{\eta\to 0+0} \int_\eta^1 x^{-\alpha}dx \overset{\alpha\ne 1}{=} \lim_{\eta\to 0+0} \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_\eta^1 =$

$= \frac{1}{1-\alpha}\left[\lim_{\eta\to 0+0} \{1 - \eta^{-\alpha+1}\}\right]$ $\xrightarrow{\alpha>1} \infty$, $\xrightarrow{\alpha<1} \frac{1}{1-\alpha}$

=> $\left|\begin{array}{l} \alpha > 1 \quad \int_0^1 \frac{1}{x^\alpha}dx \text{ е разх} \\ 0<\alpha<1, \int_0^1 \frac{1}{x^\alpha}dx \text{ е сх. и } \int_0^1 \frac{1}{x^\alpha}dx = \frac{1}{1-\alpha} \end{array}\right.$

- $\alpha = 1$, $\lim_{\eta\to 0+0} \int_0^1 \frac{1}{x}dx = \lim_{\eta\to 0+0} \ln x\Big|_\eta^1 = \lim_{\eta\to 0+0}(\ln 1 - \ln\eta) = \infty$

=> $\int_0^1 \frac{1}{x^\alpha}dx$ е разх.

- $\int_a^b \frac{1}{(b-x)^\alpha} = \begin{cases} \text{разх} & \alpha \ge 1 \\ \text{сх}, & 0<\alpha<1, \left(\frac{(b-\alpha)^{1-\alpha}}{1-\alpha}\right) \end{cases}$

Свойства на несобств. интеграл

Свойство 1| Ако н. инт. $\int_a^{+\infty} f(x)dx$ и $\int_a^{+\infty} g(x)dx$ са сх => $\int_a^{+\infty}[f(x)+g(x)]dx$ и $\int_a^{+\infty}\lambda f(x)dx\ (\lambda\in\mathbb{R})$ са сх. и: 1) $\int_a^{+\infty}[f(x)+g(x)]dx =$
$= \int_a^{+\infty} f(x)dx + \int_a^{+\infty} g(x)dx$.

2) $\int_a^{+\infty}\lambda f(x)dx = \lambda\int_a^{+\infty} f(x)dx,\ \lambda\in\mathbb{R}$

Д-во:

1) $\lim\limits_{\xi\to+\infty}\int_a^{\xi}[f(x)+g(x)]dx = \lim\limits_{\xi\to+\infty}\left[\int_a^{\xi} f(x)dx + \int_a^{\xi} g(x)dx\right] =$

$\int_a^{+\infty}[f(x)+g(x)]dx \quad = \lim\limits_{\xi\to+\infty}\int_a^{\xi} f(x)dx + \lim\limits_{\xi\to+\infty}\int_a^{\xi} g(x)dx = \int_a^{+\infty} f(x)dx + \int_a^{+\infty} g(x)dx$

инт. са сх => $\exists \lim$

2) $\lim\limits_{\xi\to+\infty}\int_a^{\xi}\lambda f(x)dx = \lim\limits_{\xi\to+\infty}\lambda\int_a^{\xi} f(x)dx = \lambda\lim\limits_{\xi\to+\infty}\int_a^{\xi} f(x)dx = \lambda\int_a^{+\infty} f(x)dx$

$\int_a^{+\infty}\lambda f(x)dx$

<u>Свойство 2</u>| Нека $f(x)$ е непр. в/у $[a,+\infty)$ и $F(x)$ е прим. ф-я на $f(x)$ в/у $[a,+\infty)$. Несобств. инт. $\int_a^{+\infty} f(x)dx$ е сх $\Leftrightarrow$ $\exists \lim\limits_{x\to+\infty} F(x) = F(+\infty)$

Д-во: $\lim\limits_{\xi\to+\infty}\int_a^{\xi} f(x)dx = \lim\limits_{\xi\to+\infty} F(x)\Big|_a^{\xi} = \lim\limits_{\xi\to+\infty}[F(\xi)-F(a)] = \lim\limits_{\xi\to+\infty}F(x) - F(a) =$
$= F(+\infty) - F(a) = F(x)\Big|_a^{+\infty}$

<u>Пример</u>: $\int_0^{+\infty}\frac{\operatorname{arctg} x}{1+x^2}dx = \int_0^{+\infty}\operatorname{arctg} x \, d\operatorname{arctg} x = \frac{\operatorname{arctg}^2 x}{2}\Big|_0^{+\infty} =$

$= \lim\limits_{x\to+\infty}\frac{\operatorname{arctg}^2 x}{2} - \frac{\operatorname{arctg}^2 0}{2} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - 0 = \frac{\pi^2}{8}$

<u>Свойство 3</u>| Нека $f(x)$ и $g(x)$ ф-и в/у $[a,+\infty)$ и имат непр. произв. $f'(x)$ и $g'(x)$ в/у $[a,+\infty)$. Ако $\exists \lim\limits_{x\to+\infty} f(x)g(x)$ и $\int_a^{+\infty} g(x)df(x)$ е сх → $\int$ е сх. $\int_a^{+\infty} f(x)g(x)$ и $\int_a^{+\infty} f(x)dg(x) =$
$= f(x)\cdot g(x)\Big|_a^{+\infty} - \int_a^{+\infty} g(x)df(x)$ (Т-ма на НЛ за инт. по части)

Д-во: $\lim\limits_{\xi\to+\infty}\int_a^{\xi} f(x)dg(x) = \lim\limits_{\xi\to+\infty}\left[f(x)g(x)\Big|_a^{\xi} - \int_a^{\xi} g(x)df(x)\right] =$
$= \lim\limits_{\xi\to+\infty}[f(\xi)g(\xi) - f(a)g(a)] - \lim\limits_{\xi\to+\infty}\int_a^{\xi} g(x)df(x) =$
$= (f(+\infty)\cdot g(+\infty) - f(a)\cdot g(a)) - \int_a^{+\infty} g(x)df(x) =$
$= f(x)g(x)\Big|_a^{+\infty} - \int_a^{+\infty} g(x)\,df(x)$

<u>Пример</u>: $\int_0^{+\infty} x e^{-x}dx =$
$= -\int_0^{+\infty} x e^{-x}d(-x) = -\int_0^{+\infty} x\,de^{-x} = -\left[xe^{-x}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-x}dx\right]$
$= -\left[x\cdot e^{-x}\Big|_0^{+\infty} + e^{-x}\Big|_0^{+\infty}\right] =$
$= -\left[\lim\limits_{x\to+\infty} x\cdot e^{-x} - 0\cdot e^{-0} + \lim\limits_{x\to+\infty} e^{-x} - e^{-0}\right] =$
$= -[0-0+0-1] = 1$

<u>Свойство 4</u>| Нека ф-та $f(x)$ е непр. в/у $[a,+\infty)$ и ф-та $x=\varphi(t)$:

1) $\varphi: [\alpha,\beta) \to [a,+\infty)$
2) $\varphi(\alpha) = a$, $\lim\limits_{t\to\beta-0}\varphi(t) = +\infty$
3) $\exists\ \varphi'(t)$ - непр. в/у $[\alpha,\beta)$

$\Rightarrow \int_a^{+\infty} f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt = \int_\alpha^\beta f(\varphi(t))\,d\varphi(t)$

Ако $\int_a \limits^{+\infty} f(x)\,dx$ е сх!

Д-во:

$$\lim_{\xi\to+\infty}\int_a^{\xi} f(x)\,dx = \lim_{\xi\to+\infty}\int_a^{\varphi(t_\xi)} f(\varphi(t))\,\varphi'(t)\,dt = *$$

$\exists\ t_\xi \in [\alpha,\beta) : \varphi(t_\xi) = \xi$

$$* = \lim_{t'\to\beta}\int_\alpha^{t'} f(\varphi(t))\cdot\varphi'(t)\,dt = \int_\alpha^{\beta} f(\varphi(t))\cdot\varphi'(t)\,dt$$

<u>Пример:</u> $\int_0^{+\infty}\frac{1}{(x^2+1)^{\frac{3}{2}}}\,dx = \int_0^{\pi/2}\frac{1}{\frac{1}{\cos^3 t}}\,dt = \underbrace{\int_0^{\pi/2}\cos t\,dt}_{\text{собств. инт.}} = *$

$x = tg\,t \quad 0 \le x < +\infty, \quad 0 \le t < \frac{\pi}{2}$

$x^2+1 = tg^2 t + 1 = \frac{1}{\cos^2 t}$

$dx = d\,tg\,t = \frac{1}{\cos^2 t}\,dt$

$* = \sin t\,\Big|_0^{\pi/2} = 1 - 0 = \underline{1}$

<u>Свойство 5</u> | Нека $f(x) \le g(x)$, $\forall x \in [a,+\infty)$ и инт. $\int_a^{+\infty} f(x)\,dx$, $\int_a^{+\infty} g(x)\,dx$ са сх $\Rightarrow \int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$

Д-во:

т.к. $f(x) \le g(x) \Rightarrow \forall a \le \xi : \int_a^{\xi} f(x)\,dx \le \int_a^{\xi} g(x)\,dx \Rightarrow$

$\lim_{\xi\to+\infty}\int_a^{+\infty} f(x)\,dx \le \lim_{\xi\to+\infty}\int_a^{\xi} g(x)\,dx \Rightarrow \int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$

~~...~~ 12. Несобствен интеграл от неотрицателна функция.

Абсолютно и условно сходящи се несобствени интеграли.

<u>Т</u> | Нека $f(x) \ge 0$ и инт. в/у $\forall$ под.инт. $[a,\xi]$, $\forall \xi \ge a$ в/у $[a,+\infty)$

Несобств. инт. $\int_a^{+\infty} f(x)\,dx$ е сх $\Leftrightarrow F(\xi) = \int_a^{\xi} f(x)\,dx$ е огр. в/у $[a,+\infty)$

При това $\int_a^{+\infty} f(x)\,dx = \sup_{\xi\in[a,+\infty)} F(\xi)$

Д-во:

<u>Лема:</u> Нека $g(x)$ е мон. раст. в/у $[a,+\infty)$

Тогава $\exists \lim_{x\to+\infty} g(x) \Leftrightarrow g(x)$ е огр. в/у $[a,+\infty)$

При това $\lim_{x\to+\infty} g(x) = \sup_{x\in[a,+\infty)} g(x)$

$F(\xi) = \int_a^{\xi} f(x)\,dx \Rightarrow$ мон. раст. в/у $[a,+\infty)$

Действ. нека $a \le \xi_1 \le \xi_2$

Разгл. $F(\xi_2) - F(\xi_1) = \int_a^{\xi_2} f(x)\,dx - \int_a^{\xi_1} f(x)\,dx = \int_{\xi_1}^{\xi_2} f(x)\,dx \overset{\text{с-ва опр. инт.}}{\ge} 0$

$\Rightarrow F(\xi_2) \ge F(\xi_1)$, т.е. $F(x)$ е м. $\uparrow$ в/у $[a,+\infty)$

$\int_a^{+\infty} f(x)\,dx$ е сх $\overset{def}{\Leftrightarrow} \exists \lim_{\xi\to+\infty} F(\xi) \Leftrightarrow F(\xi)$ е огр. в/у $[a,+\infty)$

При това $\int_a^{+\infty} f(x)\,dx = \lim_{\xi\to+\infty} F(\xi) \overset{(\text{Л})}{=} \sup_{\xi\in[a,+\infty)} F(\xi) = \sup_{\xi\in[a,+\infty)}\int_a^{\xi} f(x)\,dx$