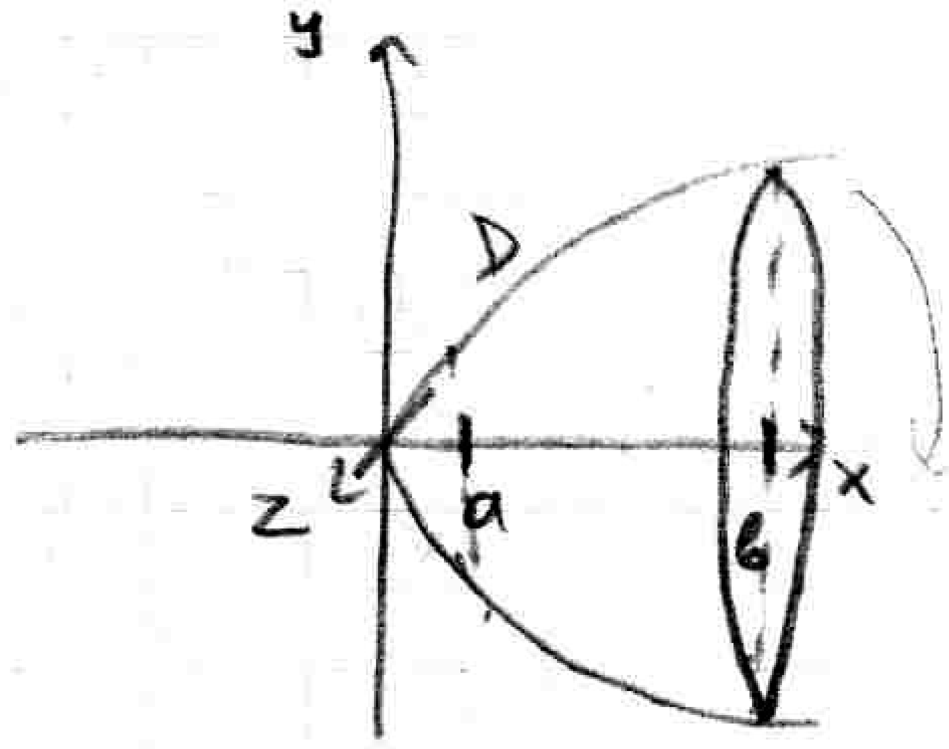
(10) Лице на ротационна повърхнина



Деф| Нека $f(x) \geq$ - непр. ф-я в/у $[a,b] \to \mathcal{P}(f)$

Нека $\tau = \{x_i\}_{i=0}^{n}$ - разб. на $[a,b]$, $\delta_\tau = \max\limits_{1\leq i\leq n} \Delta x_i$

$M_i(x_i, f(x_i))$ $i = 0 \div n$

$L_\tau = M_0 M_1 \dots M_n$ - начупена линия

$\Rightarrow \mathcal{P}(L_\tau) = \bigcup\limits_{i=1}^{n} K_i$ - ротац. повърх., обр. от L_τ, K_i - пресечен конус

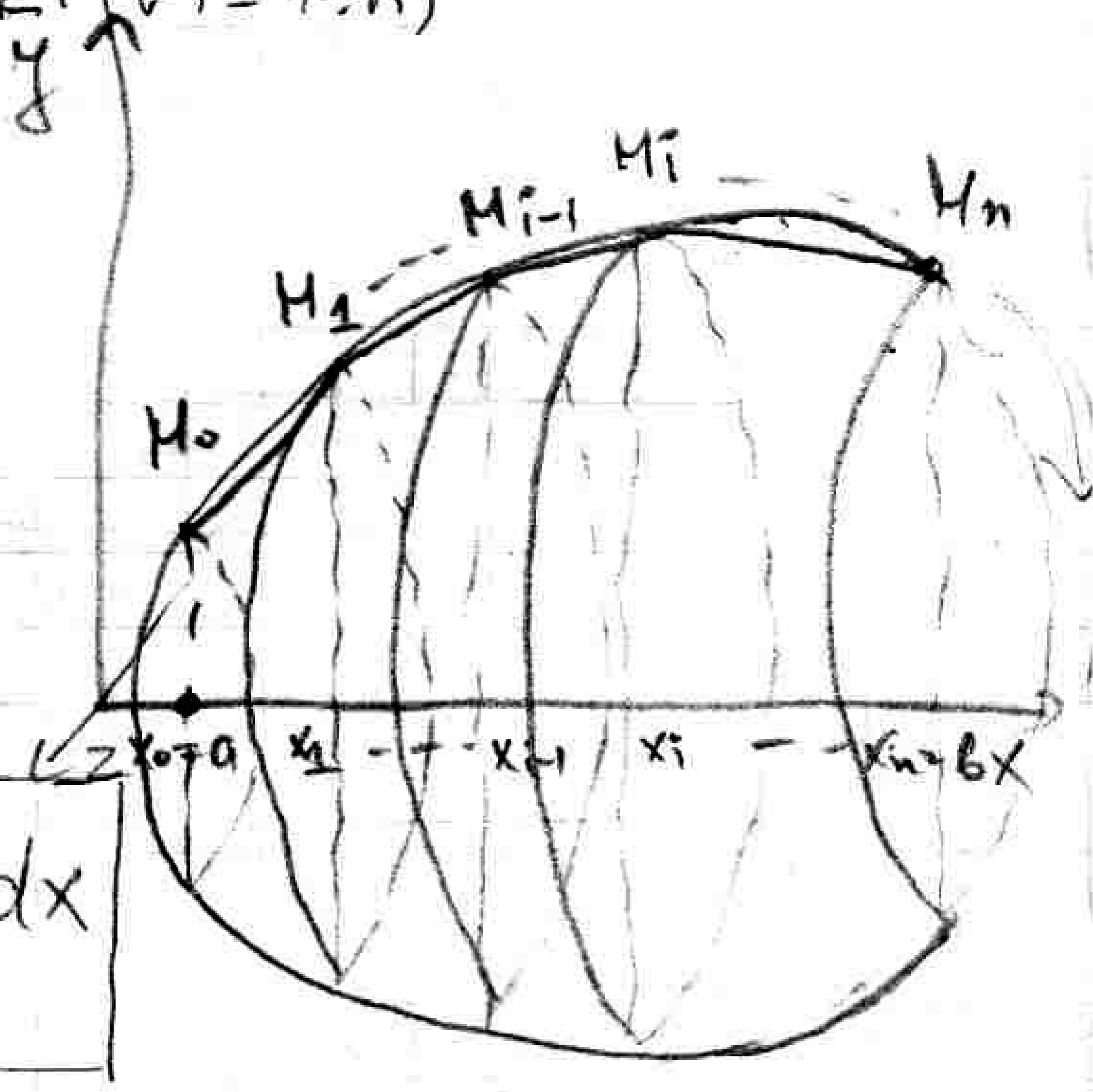
$S(\mathcal{P}(L_\tau)) = \sum\limits_{i=1}^{n} S(K_i)$, $S(K_i)$ - ок. повърх. на K_i $(\forall i = 1 \div n)$

Лице $S(\mathcal{P}(f))$ е такова число, че:

$S(\mathcal{P}(f)) = \lim\limits_{\delta_\tau \to 0} S(\mathcal{P}(L_\tau))$, т.е.

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0: \forall \tau = \{x_i\}_{i=0}^{n}, \delta_\tau < \delta$

$\Rightarrow |S(\mathcal{P}(f)) - S(\mathcal{P}(L_\tau))| < \varepsilon$



Т| Ако $f(x) \geq 0$ и има непр. пр. $f'(x)$ в/у $[a,b] \Rightarrow$ $\boxed{S(\mathcal{P}(f)) = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx}$

Д-во:

• Нека $\tau = \{x_i\}_0^n \to \mathcal{P}(L_\tau) = \bigcup\limits^{n} K_i$

$S(\mathcal{P}(L_\tau)) = \sum\limits_{i=1}^{n} S(K_i) = \sum\limits_{i=1}^{n} \pi(f(x_{i-1}) + f(x_i)) \cdot |M_{i-1}M_i| =$

$= \pi \sum\limits_{i=1}^{n} [f(x_{i-1}) - f(x_i)] \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$ (1)

$\xRightarrow{\text{Т. Лагр.}}$ $\forall i = 1 \div n$, $\exists \xi_i \in (x_{i-1}, x_i): f(x_i) - f(x_{i-1}) = f'(\xi_i) \cdot \overbrace{(x_i - x_{i-1})}^{\Delta x_i}$ (1)

$(1) = \pi \sum\limits_{i=1}^{n} [f(x_{i-1}) + f(x_i)] \sqrt{1 + [f'(\xi_i)]^2}\, \Delta x_i$

• Да разгл. ф-ята $F(x) = 2\pi f(x) \sqrt{1+[f'(x)]^2}\,\Delta x_i$ непр. в/у $[a,b] \Rightarrow$

$\exists I = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

$\Rightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \forall \tau = \{x_i\}_{i=0}^{n}, \delta_\tau < \delta, \delta = \min(\delta', \delta_1), \forall \xi = \{\xi_i\}_{i=1}^{n}$

$\Rightarrow |\sigma_\tau(F;\xi) - I| < \varepsilon$, т.е.

$\lim\limits_{\delta_\tau \to 0} \sigma_\tau(F;\xi) = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

• $\tau \to \mathcal{P}(L_\tau) \to S(\mathcal{P}(L_\tau)) = \pi \sum\limits_{i=1}^{n} [f(x_{i-1}) + f(x_i)] \sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$,

където $\xi = \{\xi_i\}_{i=1}^{n}$, $\xi_i \in (x_{i-1}, x_i)$ $\forall i = 1 \div n$

• $|I - S(\mathcal{P}(L_\tau))| = |[I - \sigma_\tau(F;\xi)] + [\sigma_\tau(F;\xi) - S(\mathcal{P}(L_\tau))]| \leq$

$\leq |I - \sigma_\tau(F;\xi)| + |\sigma_\tau(F;\xi) - S(\mathcal{P}(L_\tau))| <$

$< \varepsilon + |\sigma_\tau(F;\xi) - S(\mathcal{P}(L_\tau))|$ ★

• $|\sigma_\tau(f,\xi) - S(P(L_\tau))| = \left|\sum_{i=1}^{n} 2\pi f(\xi_i)\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i - \pi\sum_{i=1}^{n}[f(x_{i-1})+f(x_i)]\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i\right|$

$= \pi\left|\sum_{i=1}^{n}\left([\underbrace{f(\xi_i)-f(x_{i-1})}_{(2)\Rightarrow<\varepsilon}] + [\underbrace{f(\xi_i)-f(x_i)}_{(2)\Rightarrow<\varepsilon}]\right)\right|\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$

$\le \pi\sum_{i=1}^{n}\left(|f(\xi_i)-f(x_{i-1})| + |f(\xi_i)-f(x_i)|\right)\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$

• $f(x) \in$ непр. в/у $[a,b] \to f(x)$ е равном. непр $\Rightarrow$

$\varepsilon>0,\ \exists\ \delta' = \delta'(\varepsilon) > 0 : \forall\, x', x'' \in [a,b]:$

$|x'-x''| < \delta' \Rightarrow |f(x')-f(x'')| < \varepsilon$

$\left.\begin{array}{l}|x_{i-1}-\xi_i| \le \Delta x_i \le \delta_\tau < \delta \\ |x_i-\xi_i| \le \Delta x_i \le \delta_\tau < \delta\end{array}\right. \Rightarrow \left.\begin{cases}|f(x_i)-f(\xi_i)| < \varepsilon \\ |f(x_{i-1})-f(\xi_i)| < \varepsilon\end{cases}\right\}(2)$

$\bigstar < \pi\sum_{i=1}^{n} 2\varepsilon\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i\ \bigstar\bigstar$

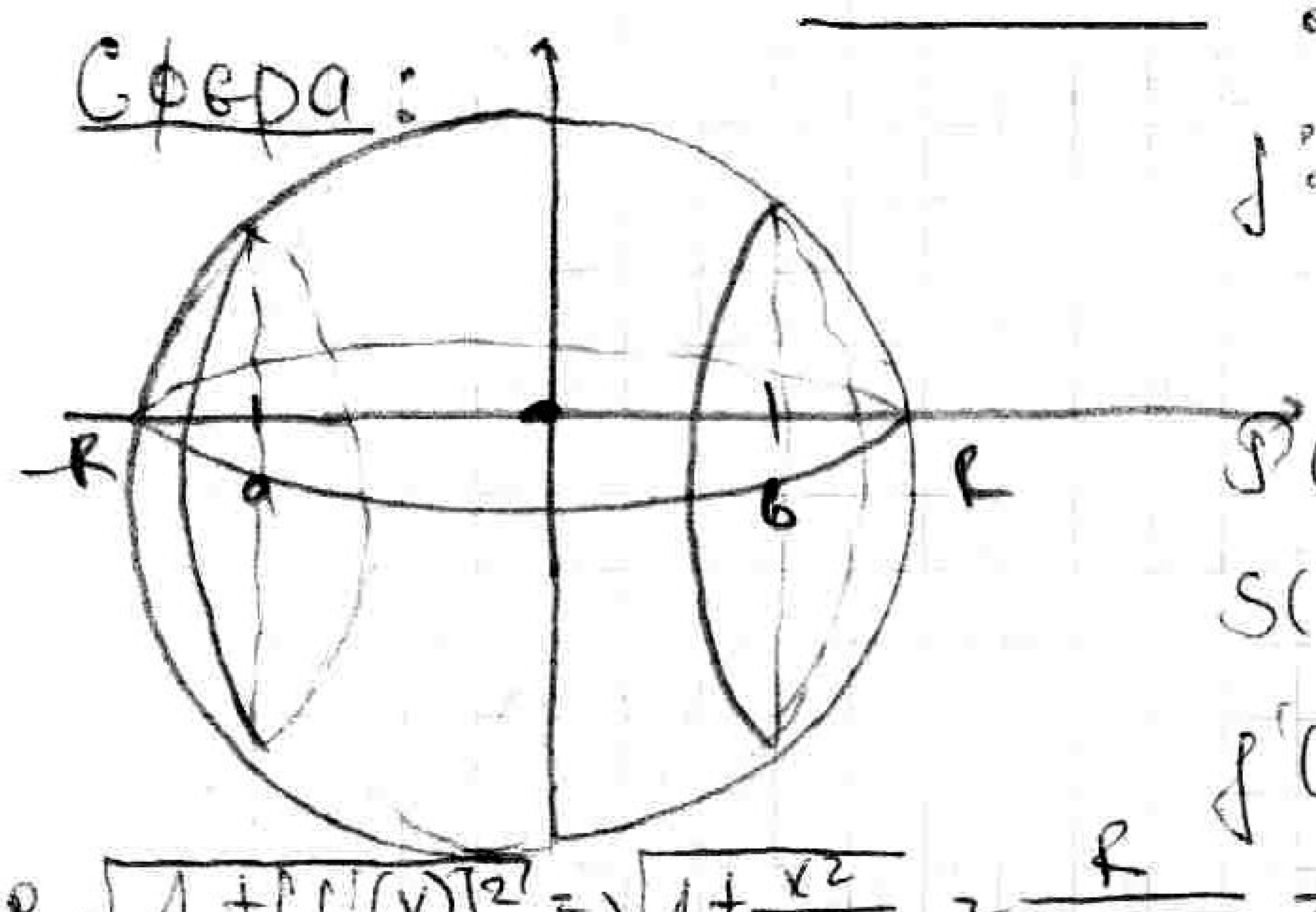
• $\sqrt{1+[f'(x)]^2} \in$ непр. в/у $[a,b] \to$ е огр в $[a,b]$, т.е. $\exists\ C>0:$

$\forall\, x \in [a,b]: \sqrt{1+[f'(x)]^2} \le C\ (3)$

$\overset{(3)}{\Rightarrow} \bigstar\bigstar < 2\pi\varepsilon C\sum_{i=1}^{n}\Delta x_i = \underbrace{2\pi(b-a)\cdot C}_{const}\cdot\varepsilon < [1+2\pi(b-a)C]\,\varepsilon$

$S(P(f)) = \lim S(P(L_\tau)) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

Сфера:



$f: x^2+y^2 = R^2 \Rightarrow y = f(x) = \sqrt{R^2-x^2},$
$x \in [a,b] \subset [-R,R]$

$P(f)$ — сфера с рад. R и ц-р $(0,0)$

$S(P(f)) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

$f'(x) = \left(\sqrt{R^2-x^2}\right)' = -\frac{2x}{2\sqrt{R^2-x^2}} = -\frac{x}{\sqrt{R^2-x^2}}$

• $\sqrt{1+[f'(x)]^2} = \sqrt{1+\frac{x^2}{R^2-x^2}} = \frac{R}{\sqrt{R^2-x^2}} \Rightarrow$

$f(x)\sqrt{1-[f'(x)]^2} = \cancel{\sqrt{R^2-x^2}}\cdot\frac{R}{\cancel{\sqrt{R^2-x^2}}} = R$

$\Rightarrow S(P(f)) = 2\pi\int_a^b R\,dx = 2\pi R(b-a)$ - лице сферичен пояс (защр)

• Ако $[a,b] = [-R,R]$ получ. лицето на сфера

$$\boxed{S_{сф} = 4\pi R^2}$$

(11) Несобствени интеграли в/у безкраен интервал и от неогр. ф-я - опр., св-ва.

Примери: 1)

$f(x) = \frac{1}{1+x^2},\ x \in [0,+\infty)$

$D = \{(x,y): x \ge 0,\ 0 \le y \le \frac{1}{1+x^2}\}$ - неогр. м-во защр.

$S(D) = ?$

$0 < \xi \to$ произв. число