$D_i = \Omega(\xi_i) \Rightarrow ц(D_i) = D_i \times [x_{i-1}, x_i]$

$P_i = \Omega(\eta_i) \Rightarrow ц(P_i) = P_i \times [x_{i-1}, x_i]$

$\bigcup_{i=1}^{n} ц^{изм}(D_i) \subset \Omega \subseteq \bigcup_{i=1}^{n} ц^{изм}(D_i)$

$V(\bigcup_{i=1}^{n} ц(D_i)) = \sum_{i=1}^{n} V(ц(D_i)) = \sum_{i=1}^{n} лиц(D_i)(x_{i-1}, x_i) = \sum_{i=1}^{n} S(\xi_i)\Delta x_i = \sum_{i=1}^{n} m_i \Delta x_i = s_\tau(S(x))$

$V(\bigcup_{i=1}^{n} ц(P_i)) = \sum_{i=1}^{n} V(ц(P_i)) = \sum_{i=1}^{n} лиц(P_i)(x_{i-1}, x_i) = \sum_{i=1}^{n} S(\eta_i)\Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i = S_\tau(S(x))$

$S(x)$ – непр. в/у $[a,b] \Rightarrow S(x)$ – инт. $\Rightarrow$

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=0}^{n}, \delta_\tau < \delta \Rightarrow S_\tau(S(x)) - s_\tau(S(x)) < \varepsilon$

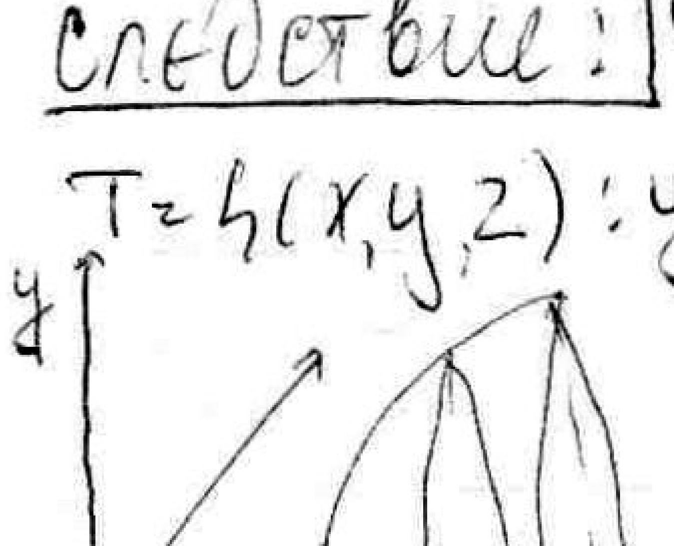
$\Rightarrow V(\bigcup_{i=1}^{n} ц(P_i)) - V(\bigcup_{i=1}^{n} ц(D_i)) = S(S(x)) - s(S(x)) < \varepsilon$

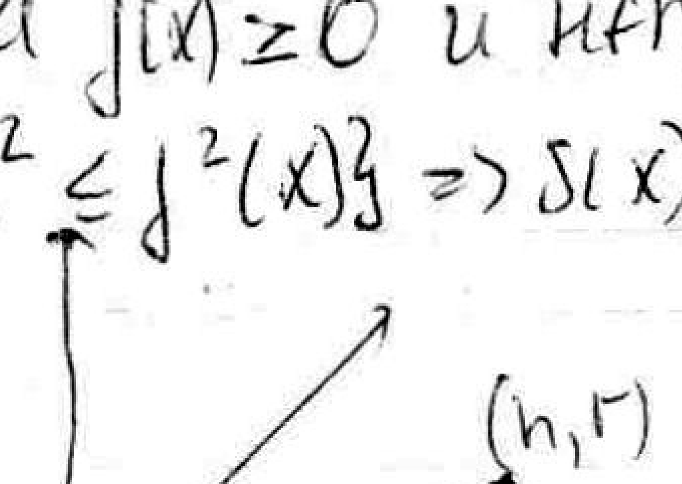
$\Rightarrow \Omega$ – изм. тяло в $\mathbb{R}^3$

$$V(\Omega) = \sup_\tau V(\bigcup_{i=1}^{n} ц(D_i)) = \sup_\tau V(\bigcup_{i=1}^{n} ц(P_i)) = \int_a^b S(x)\,dx$$

Следствие: Нека $f(x) \geq 0$ и непр. в/у $[a,b]$

$T = \{(x,y,z) : y^2 + z^2 \leq f^2(x)\} \Rightarrow S(x) = \pi f^2(x)$ и $V(T) = \int_a^b \pi f^2(x)\,dx = \pi \int_a^b f^2(x)\,dx$

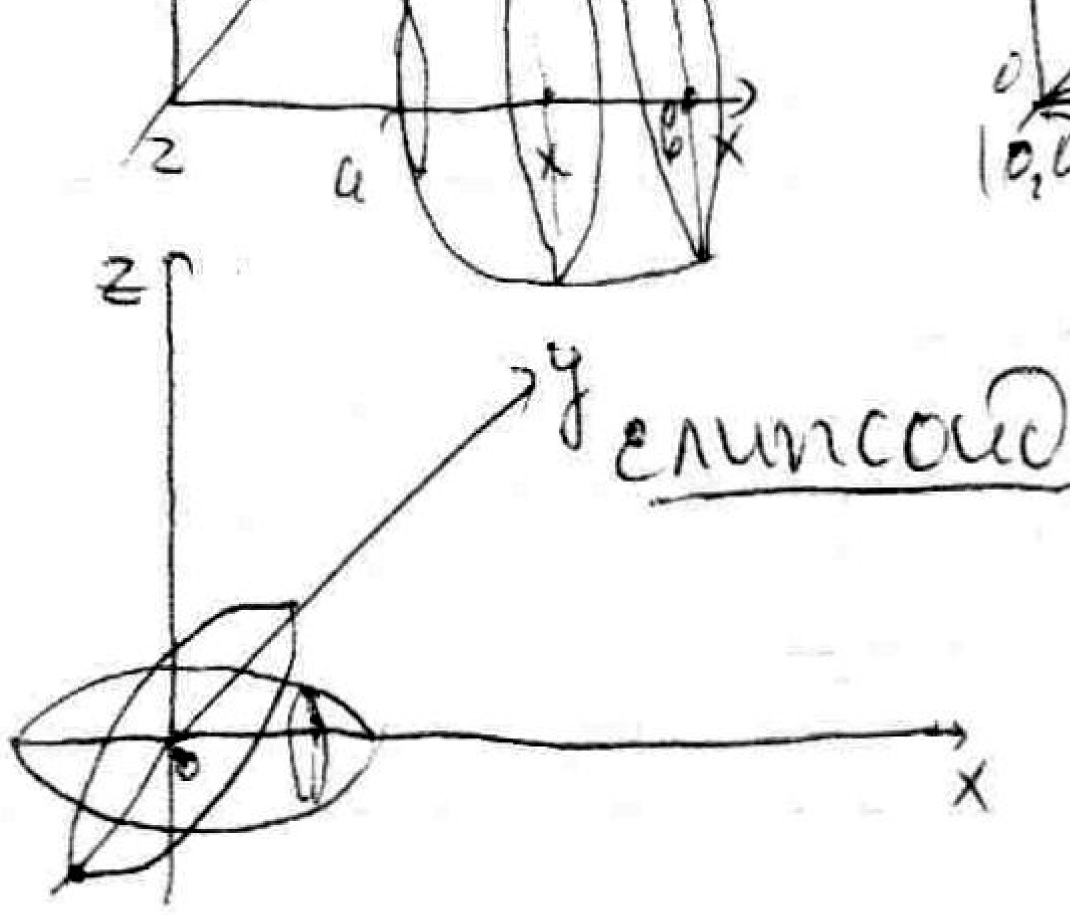




$\ell : y = \frac{r}{h} x$

$\Rightarrow V(конус) = \pi \int_0^h \left(\frac{r}{h} x\right)^2 dx =$

$= \frac{\pi r^2}{h^2} \int_0^h x^2\,dx = \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3}\Big|_0^h = \frac{\pi r^2}{\cancel{h^2}} \cdot \frac{h^3}{3} =$



елипсоид:

$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$x \in [-a, a]$

$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} \;\Big|\; : \left(1 - \frac{x^2}{a^2}\right)$

$E_x : \frac{y^2}{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2} + \frac{z^2}{\left(c\sqrt{1-\frac{x^2}{a^2}}\right)^2} = 1$

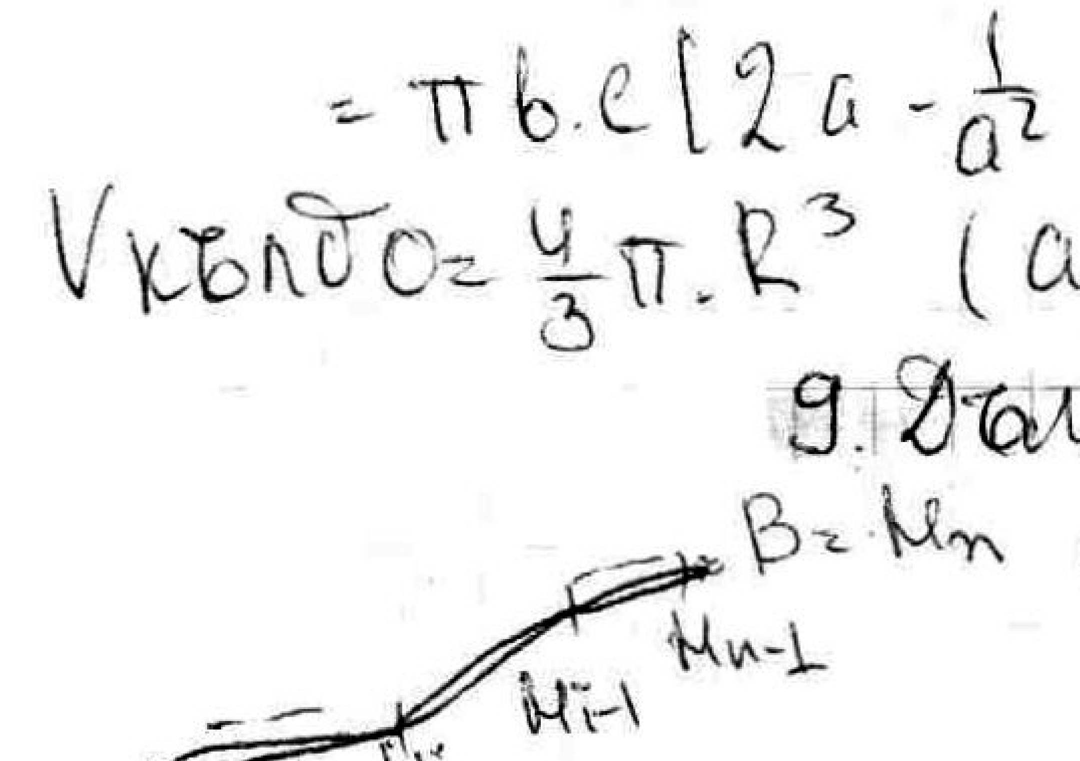
$S(x) = S(E_x) = \pi b c \left(1 - \frac{x^2}{a^2}\right)$

$V(E) = \int_{-a}^{a} \pi . b . c \left(1 - \frac{x^2}{a^2}\right) dx = \pi b c \left[\int_{-a}^{a} 1\,dx - \frac{1}{a^2}\int_{-a}^{a} x^2\,dx\right] =$

$= \pi b . c \left[2a - \frac{1}{a^2} \cdot \frac{x^3}{3}\Big|_{-a}^{a}\right] = \pi b c \left[2a - \frac{1}{\cancel{a^2}} \cdot \frac{2}{3} \cdot a^{\cancel{3}}\right] = \pi b c . \frac{4}{3} a$

$V_{кълбо} = \frac{4}{3} \pi . R^3 \quad (a = b = c = R)$

9. Дължина на крива линия.



$\tau : \{M_i\}_{i=0}^{n} \quad M_i \in \ell$

$M_0 = A, \; M_n = B$

$L(M_0 \ldots M_n) = \bigcup_{i=1}^{n} [M_{i-1}, M_i]$ – вписана начупена линия в ℓ

$d(L) =$ дълж. на $(L(M_0 \ldots M_n)) = \sum_{i=1}^{n} |M_{i-1}, M_i|$

$\Delta_\tau = \max |M_{i-1}, M_i|$ – пар. на разб.

$\delta_\tau = \max \Delta x_i, \; i = 1 \div n$

$0 < \delta_\tau \leq \Delta_\tau$

Деф Дължина на кр. л. l, наричаме число $d(l)$:

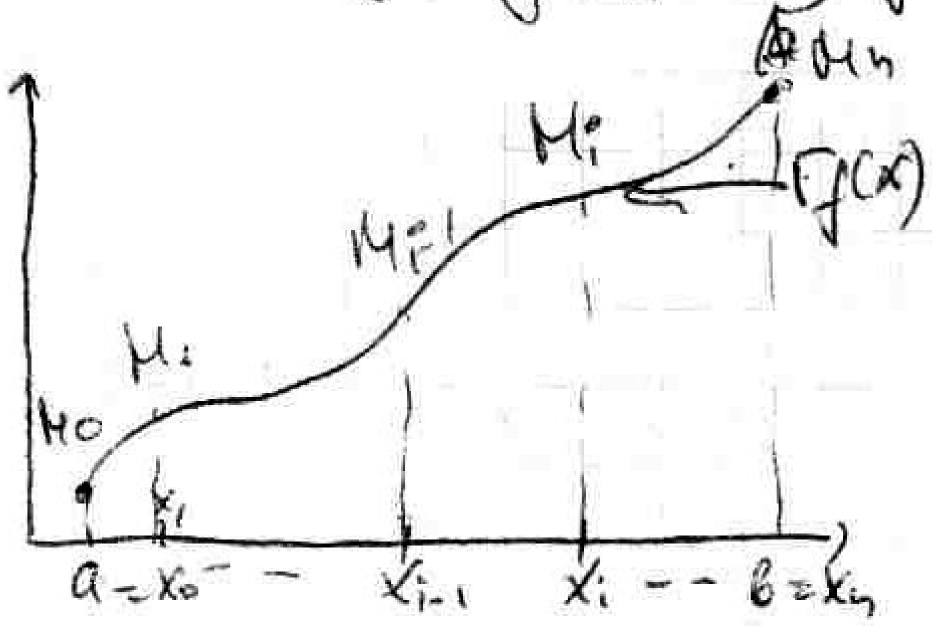
$$\forall \varepsilon>0,\ \exists \delta=\delta(\varepsilon)>0 : \forall \tau=\{M_i\}_{i=0}^{n},\ \Delta_\tau<\delta$$

$$\Rightarrow |d(l)-d(L(M_0 \dots M_n))|<\varepsilon$$

т.е. $d(l)=\lim\limits_{\delta\to 0} d(L(M_0 \dots M_n))$

Т1 Нека $f(x) \in$ диф в/у $[a,b]$ и $f'(x) \in$ непр в/у $[a,b]$

$$\Rightarrow d(\Gamma f(x))=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$



Д-во:

Нека $T=\{M_i\}_{i=0}^{n}$ на $\Gamma f(x)$, $i=0\div n$

Нека $M\{(x_i, f(x_i))\}_{i=0}^{n} \Rightarrow \tau=\{x_i\}_{i=0}^{n}$ р. на $[a,b]$

$L(M_0 \dots M_n) \Rightarrow d(L(M_0 \dots M_n))=\sum_{i=1}^{n}|M_{i-1}, M_i|$

$$|M_{i-1}, M_i|=\sqrt{(x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))^2} \ (*)\quad \forall i=1\div n$$

Разгл. f в/у $[x_{i-1}, x_i]$: $\overset{\text{Т.Л.}}{\Rightarrow} \exists \xi_i \in [x_{i-1}, x_i] : f(x_i)-f(x_{i-1})=f'(\xi_i).(x_i-x_{i-1})$

$$\overset{(*)}{=} \sqrt{(x_i-x_{i-1})^2+(f'(\xi_i))^2(x_i-x_{i-1})^2}=\sqrt{1+(f'(\xi_i))^2}\cdot(\Delta x_i) \Rightarrow$$

$\sum_{i=1}^{n}\sqrt{1+[f'(\xi_i)]^2}\cdot\Delta x_i$, където $\xi=\{\xi_i\}_{i=1}^{n} : \xi_i \in [x_{i-1}, x_i]$ за $\forall i=1\div n$

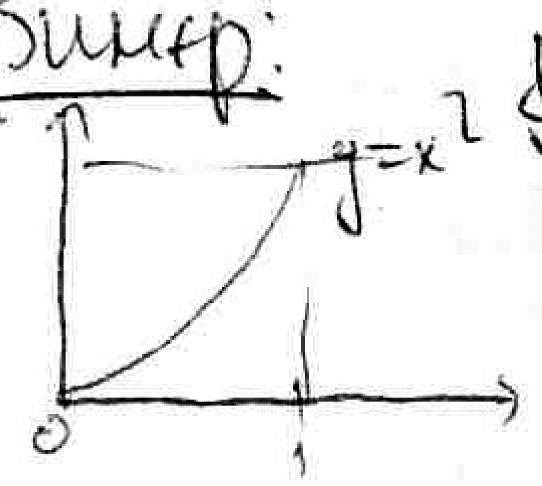
$\sigma_\tau(\sqrt{1+[f'(x)]^2}; \xi)$

$$\lim d(L(M_0 \dots M_n))=\lim_{\Delta_\tau\to 0}\sum_{i=1}^{n}\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i=\lim_{\delta_\tau\to 0}\sum_{i=1}^{n}\sqrt{1+[f'(x_0)]^2}\,\Delta x_i=$$

$$=\lim_{\delta_\tau\to 0}\sigma_\tau(\sqrt{1+[f'(x)]^2}; \xi)=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$$

$$\underset{0}{\underset{\downarrow}{0}} \le \underset{0}{\underset{\downarrow}{\delta_\tau}} < \underset{0}{\underset{\downarrow}{\Delta_\tau}}$$

Пример:



$f(x)=x^2$, $[0,1]$

$$y=x^2,\quad d(\Gamma x^2)=\int_0^1\sqrt{1+[(x^2)']^2}\,dx=\int_0^1\sqrt{1+(2x)^2}\,dx=\int_0^1\sqrt{1+(2x)^2}\,dx=$$

$$=\frac{1}{2}\int\sqrt{1+(2x)^2}\,d(2x)\overset{2x=t}{=}\frac{1}{2}\int_0^2\sqrt{1+t^2}\,dt$$

$$I=\int\sqrt{1+t^2}\,dt=t\sqrt{1+t^2}-\int t\,d\sqrt{1+t^2}=t\sqrt{1+t^2}-\int\frac{t^2+1-1}{\sqrt{1+t^2}}\,dt$$

$$=t\sqrt{1+t^2}-\int\sqrt{1+t^2}\,dt+\int\frac{1}{\sqrt{1+t^2}}\,dt=t\sqrt{1+t^2}-I+\ln(t+\sqrt{1+t^2})+C$$

$$I=\frac{1}{2}\left(t\sqrt{1+t^2}+\ln(t+\sqrt{1+t^2})\right)+C$$

$$\frac{1}{2}\cdot\frac{1}{2}\left(t\sqrt{1+t^2}+\ln(t+\sqrt{1+t^2})\right)\Big|_0^2=\frac{\sqrt{5}}{2}\ln\sqrt[4]{2+\sqrt{5}}$$