Деф| Окр.(R): $\rho = \rho(\theta) = R$, $0 \le \theta \le 2\pi$

$B_R(\theta): \{(\theta, \rho): 0 \le \theta \le 2\pi,\ 0 \le \rho \le R\}$

$$S(B_R(0)) = \frac{1}{2}\int_0^{2\pi} R^2\, d\theta = \frac{1}{2}R^2 \cdot \int_0^{2\pi} 1\, d\theta = R^2\, 2\pi\, \frac{1}{2} = \pi R^2$$

- $\rho = a(1+\cos\theta)$, $0 \le \theta \le 2\pi$

$$S(G) = \frac{1}{2}\int_0^{2\pi} a^2(1+\cos\theta)^2\, d\theta = \frac{a^2}{2}\int_0^{2\pi}(1+\cos\theta+\cos^2\theta)\, d\theta =$$

$$= \frac{a^2}{2}\left[\int_0^{2\pi} 1\, d\theta + 2\int_0^{2\pi}\cos\theta\, d\theta + \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\, d\theta\right] =$$

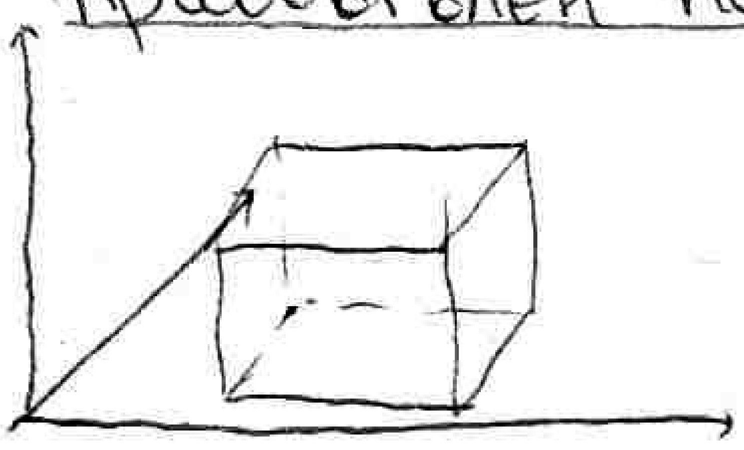
$$= \frac{a^2}{2}\left[2\pi + 2\sin\theta\Big|_0^{2\pi} + \frac{1}{2}\left(2\pi + \frac{1}{2}\sin 2\theta\Big|_0^{2\pi}\right)\right] =$$

$$= \frac{a^2}{2}(2\pi + \pi) = \frac{3}{2}\pi a^2$$

8. Обем на тяло с известно напречно сечение. Обем на ротационно тяло

Деф| $\Pi = \langle a_1, b_1\rangle \times \langle a_2, b_2\rangle \times \langle a_3, b_3\rangle$, където $a_i, b_i \in \mathbb{R}$, $i = 1,2,3$, $\langle \in \{<, \le\}$, $\rangle \in \{>, \ge\}$

правоъгълен паралелепипед



Вътрешност на Π: $\Pi^0 = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$

Деф| Клетъчно тяло наричаме $\forall$ мн. $K = \bigcup_{i=1}^{n} \Pi_i$: $\Pi_i^0 \cap \Pi_j^0 = \emptyset$ $(\forall i, j = 1 \div n,\ i \ne j)$

Деф| $V(\Pi) = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$ – обем на Π

$V(K) = \sum_{i=1}^{n} V(\Pi_i)$ – обем на кл. тяло

Деф| Нека $\Omega \subset \mathbb{R}^3$. Казваме, че Ω е измеримо м-во/тяло, ако за $\forall \varepsilon > 0$, $\exists$ кл. тяло k, K: 1) $k \subset \Omega \subset K$
2) $V(K) - V(k) < \varepsilon$

Деф| Нека Ω е измеримо тяло. Обем на Ω се нарича такова число $V(\Omega)$: $\forall k, K$ (кл. тела): $k \subset \Omega \subset K \Rightarrow$
$V(k) \le V(\Omega) \le V(K)$

Т| Ако Ω е измеримо тяло в $\mathbb{R}^3$, то $\exists!\ V(\Omega)$ на Ω. При това

$$V(\Omega) = \sup_{k \subset \Omega} V(k) = \inf_{\Omega \subset K} V(K)$$

Д-во:

Нека Ω – измеримо м-во в $\mathbb{R}^3$

$\forall$ клетъчно м-во k, K: $k \subset \Omega \subset K \Rightarrow V(k) \le V(K)$

$\Rightarrow \exists\ V(k) \le \sup_{k \subset \Omega} V(k) \le V(K)$

$\Rightarrow \exists \inf_{\Omega \subset K} V(K) \Rightarrow V(k) \le \sup V(k) \le \inf V(K) \le V(K)$ за $\forall k, K$ $k \subset \Omega \subset K$

$\Rightarrow$ Ако $V(\Omega)$ е обем на $\Omega \Rightarrow V(\Omega) \le \inf_{\Omega \subset K} V(K)$

$$(+)\begin{cases} \inf_{\Omega \subset K} V(K) \le V(K) \\ -\sup V(k) \le -V(k) \end{cases}$$

$\Rightarrow \inf\limits_{\Omega\subset\mathcal{K}} V(\mathcal{K}) - \sup\limits_{K\subset\Omega} V(K) \le V(\mathcal{K}) - V(K) \le V(\mathcal{K}_\varepsilon) - V(K_\varepsilon) < \varepsilon$

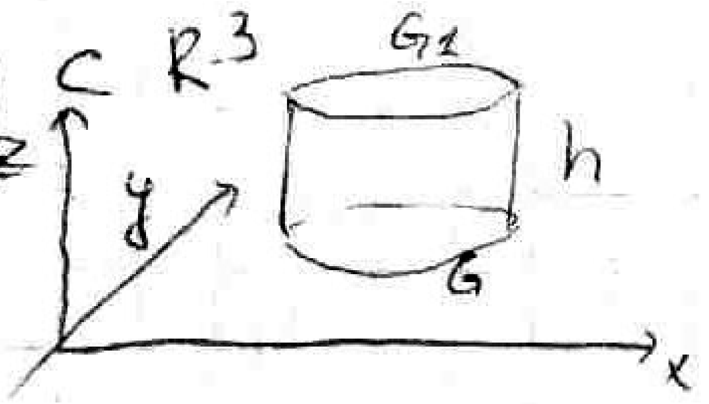
Ω - измеримо м-во $\Rightarrow \forall \varepsilon>0: \exists K, \mathcal{K}:$ 1) $K_\varepsilon \subset \Omega \subset \mathcal{K}_\varepsilon$
2) $V(\mathcal{K}_\varepsilon) - V(K_\varepsilon) < \varepsilon$

$\Rightarrow \inf\limits_{\Omega\subset\mathcal{K}} V(\mathcal{K}) - \sup\limits_{K\subset\Omega} V(K) = 0$

$\Rightarrow \underbrace{\inf\limits_{\Omega\subset\mathcal{K}} V(\mathcal{K}) = \sup\limits_{K\subset\Omega} V(K)}_{V'(\Omega)} \Rightarrow$ лицето е !

Т) Тялото Ω е измеримо $\Leftrightarrow$ за $\forall \varepsilon>0$, $\exists$ изм. тела : E, F:
1) $E \subset \Omega \subset F$
2) $V(F) - V(E) < \varepsilon$

Деф) Нека $G\subset R^2$, $h>0$
Множ. $Ц(G) = \{(x,y,z) : (x,y)\in G, 0\le z\le h\} \subset R^3$
се нарича цилиндър с осн. G
$G = \{(x,y,z) : (x,y)\in G, z=0\}$ } осн. на
$G_1 = \{(x,y,z) : (x,y)\in G, z=h\}$ } цилиндъра



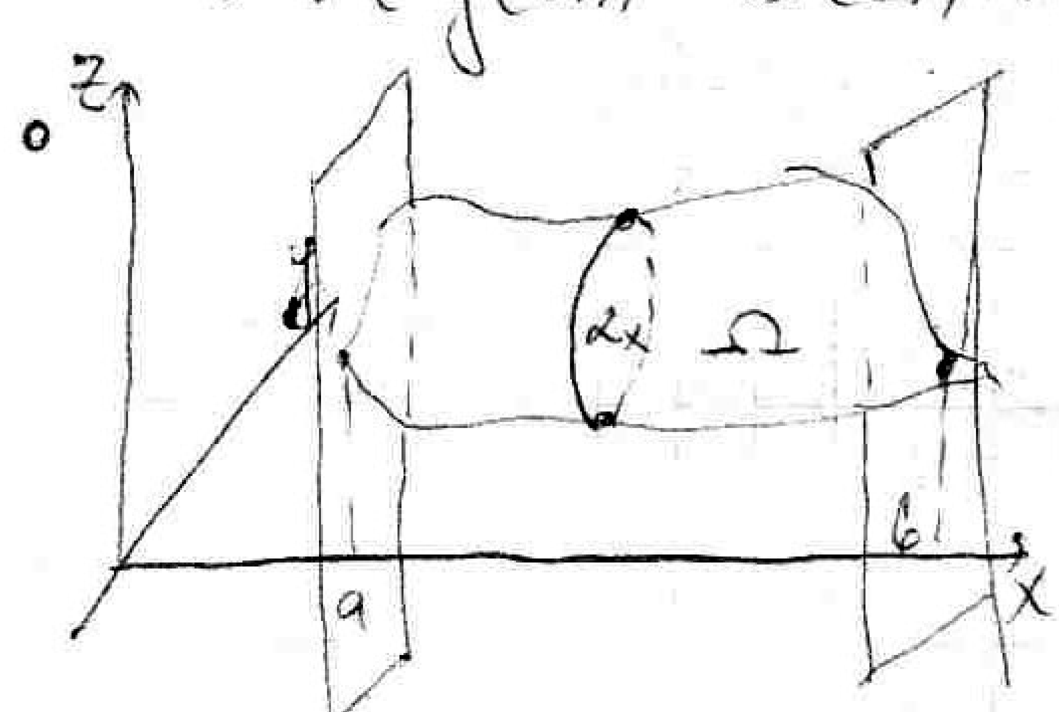
Т) Нека $G\subset R^2$ е измеримо мн, $h>0$
$\Rightarrow Ц(G) = \{(x,y,z) : (x,y)\in G, 0\le z\le h\}$ е изм. м-во и обема
$\boxed{V(Ц(G)) = S(G).h}$

Д-во:

(1) G - изм. м-во $\Rightarrow \forall \varepsilon>0$, $\exists$ кл. мн-ва K и $\mathcal{K}$:
1) $K\subset G\subset \mathcal{K}$
2) $S(\mathcal{K}) - S(K) < \varepsilon/h$
$\Pi(K) = K\times[0,h]$ - кл. тяло в R^3
$\Pi(\mathcal{K}) = \mathcal{K}\times[0,h]$ - кл. тяло в R^3
т.к. $K\subset G\subset \mathcal{K} \Rightarrow K\times[0,h] \subset G\times[0,h] \subset \mathcal{K}\times[0,h]$
$\Pi(K) \subset Ц(G) \subset \Pi(\mathcal{K})$ - кл. тела
$V(\Pi(\mathcal{K}) - \Pi(K)) = (S(\mathcal{K}) - S(K))\, h < \frac{\varepsilon}{h}\cdot h$
$\Rightarrow Ц(G)$ е измеримо м-во

(2) $V(Ц(G)) = \sup\limits_{P\subset Ц(G)} V(P) = \sup\limits_{K\subset G} S(K)\,.\,h = h.\sup\limits_{K\subset G} S(K) = \underline{h.\,S(G)}$
$P = K\times[0,h]$ - K - кл. м-во, $K\in R^2$

$\Rightarrow V(Ц(G)) = S(G).h$

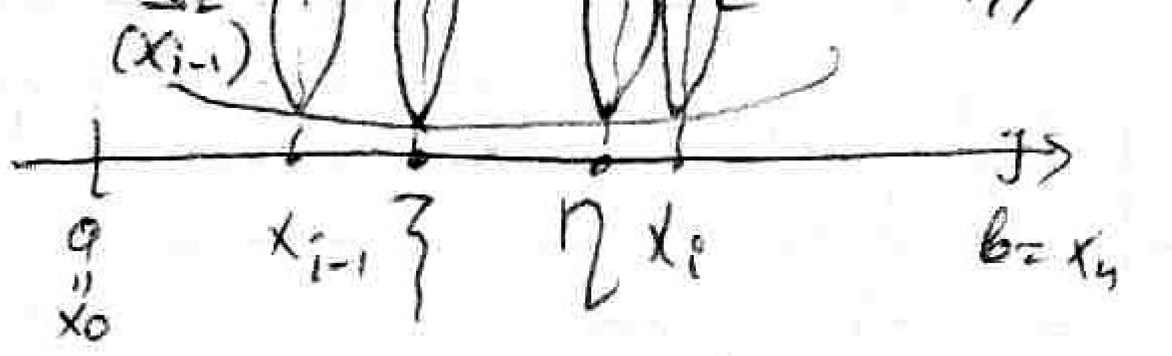


$x\in \vec{Ox} \Rightarrow \alpha_x \ni x : \alpha_x \perp \vec{Ox}$
$(\alpha_x \parallel yOz)$
$\Omega\subset R^3$, $\forall (x,y,z)\in\Omega: a\le x\le b$
$\Rightarrow \forall x\in[a,b] \Leftrightarrow \alpha_x \cap \Omega \ne \emptyset$
$\Omega(x) = \alpha_x \cap \Omega$
$\forall x\in[a,b]$: $\Omega(x)$ е измеримо и $S(x) =$ лице$(\Omega(x))$
$S[a,b]\to R$
$S = S(x)$ - непр.
$\forall x,y\in[a,b]$: $Пр_{yOz}\,\Omega(x) \subseteq Пр_{yOz}\,\Omega(y)$

Т) при горните предп. имаме: 1) Ω е изм. 2) $V(\Omega) = \int_a^b S(x)\,dx$

Д-во:

$\tau = \{x_i\}_{i=0}^n$ - разб. на $[a,b]$
$m_i = \inf\limits_{x\in[x_{i-1},x_i]} S(x)$, $M_i = \sup\limits_{x\in[x_{i-1},x_i]} S(x) \Rightarrow$
$\exists \xi_i\in[x_{i-1},x_i]: S(\xi_i) = m_i$
$\exists \eta\in[x_{i-1},x_i]: S(\eta_i) = M_i$ } $\forall i = 1\div n$



$D_i = \Omega(\xi_i) \Rightarrow Ц(D_i) = D_i \times [x_{i-1}, x_i]$
$P_i = \Omega(\eta_i) \Rightarrow Ц(P_i) = P_i \times [x_{i-1}, x_i]$

$\bigcup_{i=1}^{n} Ц^{изм}(D_i) \subset \Omega \subseteq \bigcup_{i=1}^{n} Ц^{изм}(P_i)$

$V(\bigcup_{i=1}^{n} Ц(D_i)) = \sum_{i=1}^{n} V(Ц(D_i)) = \sum_{i=1}^{n} лиц(D_i)(x_{i-1}, x_i) = \sum_{i=1}^{n} S(\xi_i)\Delta x_i = \sum_{i=1}^{n} m_i \Delta x_i = S_\tau(S(x))$

$V(\bigcup_{i=1}^{n} Ц(P_i)) = \sum_{i=1}^{n} V(Ц(P_i)) = \sum_{i=1}^{n} лиц(P_i)(x_{i-1}, x_i) = \sum_{i=1}^{n} S(\eta_i)\Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i = \mathcal{S}_\tau(S(x))$

$S(x)$ – непр. в/у $[a,b] \Rightarrow S(x)$ – инт. $\Rightarrow$

$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=0}^{n},\ \delta_\tau < \delta \Rightarrow \mathcal{S}_\tau(S(x)) - S_\tau(S(x)) < \varepsilon$

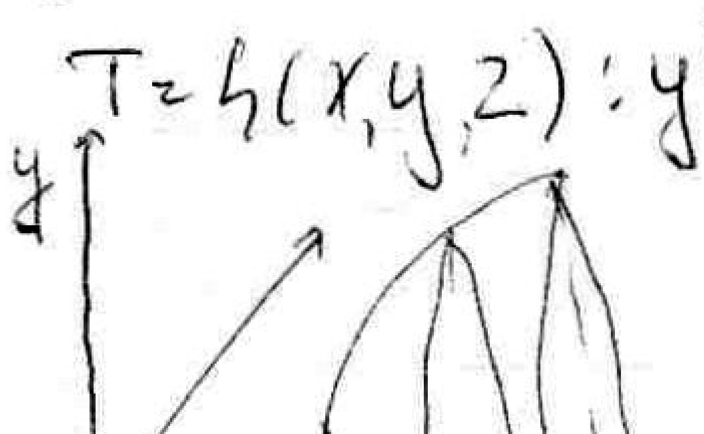
$\Rightarrow V(\bigcup_{i=1}^{n} Ц(P_i)) - V(\bigcup_{i=1}^{n} Ц(D_i)) = \mathcal{S}(S(x)) - S(S(x)) < \varepsilon$

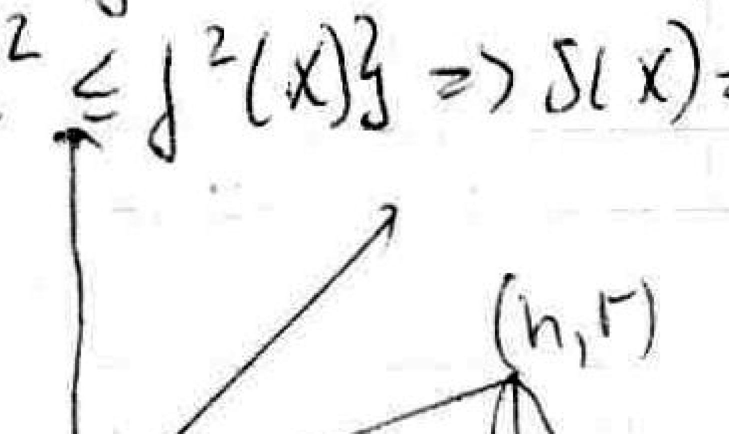
$\Rightarrow \Omega$ – изм. тяло в $\mathbb{R}^3$

$$V(\Omega) = \sup_\tau V(\bigcup_{i=1}^{n} Ц(D_i)) = \sup_\tau V(\bigcup_{i=1}^{n} Ц(P_i)) = \int_a^b S(x)\,dx$$

<u>Следствие:</u> Нека $f(x) \geq 0$ и непр. в/у $[a,b]$

$T = \{(x,y,z) : y^2 + z^2 \leq f^2(x)\} \Rightarrow S(x) = \pi f^2(x)$ и $V(T) = \int_a^b \pi f^2(x)\,dx = \pi \int_a^b f^2(x)\,dx$

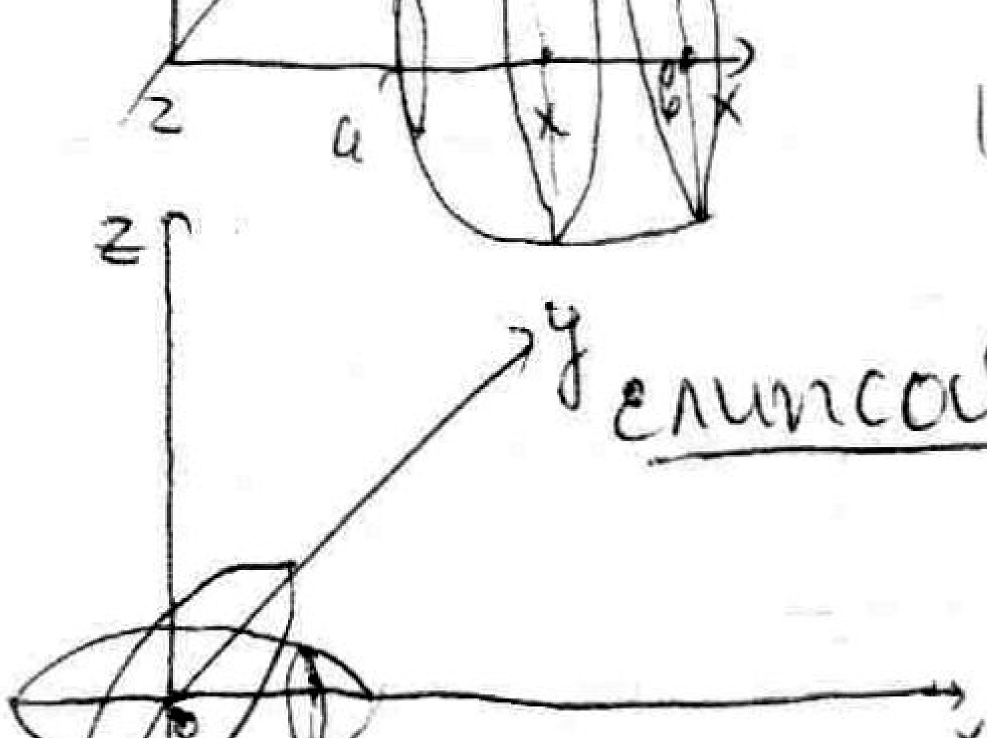




$l : y = \frac{r}{h} x$

$\Rightarrow V(конус) = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx =$

$= \frac{\pi r^2}{h^2} \int_0^h x^2\,dx = \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3}\Big|_0^h = \frac{\pi r^2}{\cancel{h^2}} \cdot \frac{h^{\cancel{3}}}{3} = \frac{B \cdot h}{3}$



<u>елипсоид:</u>

$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$x \in [-a, a]$

$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} \Big| : \left(1 - \frac{x^2}{a^2}\right)$

$\mathcal{E}_x : \frac{y^2}{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2} + \frac{z^2}{\left(c\sqrt{1-\frac{x^2}{a^2}}\right)^2} = 1$

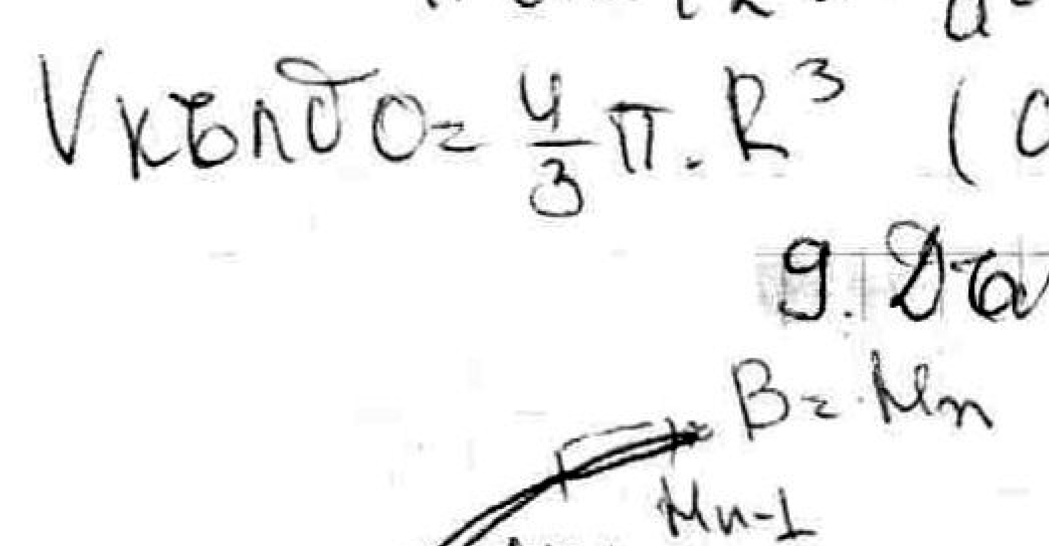
$S(x) = S(\mathcal{E}_x) = \pi b c \left(1 - \frac{x^2}{a^2}\right)$

$V(\mathcal{E}) = \int_{-a}^{a} \pi . b . c \left(1 - \frac{x^2}{a^2}\right) dx = \pi b c \left[\int_{-a}^{a} 1\,dx - \frac{1}{a^2}\int_{-a}^{a} x^2\,dx\right] =$

$= \pi b . c \left[2a - \frac{1}{a^2} \cdot \frac{x^3}{3}\Big|_{-a}^{a}\right] = \pi b c \left[2a - \frac{1}{\cancel{a^2}} \cdot \frac{2}{3} \cdot a^{\cancel{3}}\right] = \pi b c . \frac{4}{3} a$

$V_{кълбо} = \frac{4}{3}\pi . R^3 \quad (a = b = c = R)$

9. Дължина на крива линия.



$\tau : \{M_i\}_{i=0}^{n} \quad M_i \in l$

$M_0 = A,\ M_n = B$

$L(M_0 \ldots M_n) = \bigcup_{i=1}^{n} [M_{i-1}, M_i]$ – вписана начупена линия в l

$d(L) =$ дълж. на $(L(M_0 \ldots M_n)) = \sum_{i=1}^{n} |M_{i-1}, M_i|$

$\Delta_\tau = \max |M_{i-1}, M_i|$ – поп. на разб.

$\delta_\tau = \max \Delta x_i,\ i = 1 \div n$

$0 \leq \delta_\tau \leq \Delta_\tau$