$f(x).g(x)$ – има непр. производна в/у $[a,b]$, т.к. $(f(x).g(x))' = f'(x)g(x) + f(x)g'(x)$ $\forall x \in [a,b]$

$\Rightarrow \int_a^b (f(x)g(x))'dx = \int_a^b [f'(x)g(x) + g'(x)f(x)]dx = \int_a^b f'(x)g(x)dx + \int_a^b g'(x)f(x) =$

$= \int_a^b g(x)\,df(x) + \int_a^b f(x)\,dg(x)$

$\Rightarrow \int_a^b (f(x).g(x))'dx = f(x).g(x)\Big|_a^b - \int_a^b g(x)\,df(x)$

Примери 1) $\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,d\sin t = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t}\cos t\,dt = \int_0^{\frac{\pi}{2}} |\cos t|\cos t\,dt =$

$= \int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}dt = \frac{1}{2}\left[\int_0^{\pi/2} 1\,dt + \int_0^{\frac{\pi}{2}} \cos 2t\,dt\right] =$

$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2t\,d2t\right] = \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{2}}\right] = \frac{1}{2}\left[\frac{\pi}{2} + \underbrace{\frac{1}{2}\sin\frac{2\pi}{2} - \frac{1}{2}\sin\frac{2\cdot 0}{2}}_{0}\right]$

$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$, където $x = \sin t$

$x = \varphi(t) = \sin t$, $t \in [0, \frac{\pi}{2}]$, $\alpha = 0 \Rightarrow \varphi(\alpha) = \varphi(0) = \sin 0 = 0$

$\beta = \frac{\pi}{2} \Rightarrow \varphi(\beta) = \varphi(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$\varphi(t) = \sin t$ има непр. пр. в/у $[0, \frac{\pi}{2}]$

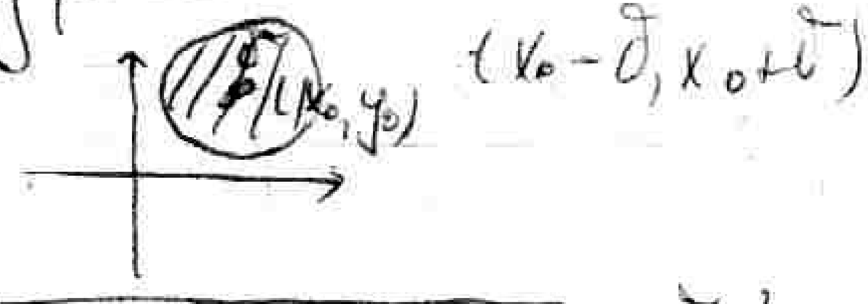
2) $\int_1^2 x\ln x\,dx = \int_1^2 \ln x\,d(\frac{x^2}{2}) = \frac{1}{2}\int_1^2 \ln x\,dx^2 = \frac{1}{2}\left[x^2\ln x\Big|_1^2 - \int_1^2 x^2 \ln x\right] =$

$= \frac{1}{2}\left[(4\ln 2 - \ln 1) - \int_1^2 x^2\,d\ln x\right] = \frac{1}{2}\left[(4\ln 2 - \ln 1) - \int_1^2 x^2\frac{1}{x}.dx\right] =$

$= \frac{1}{2}\left[4\ln 2 - \ln 1 - \frac{x^2}{2}\Big|_1^2\right] = \frac{1}{2}\left[4\ln 2 - \frac{3}{2}\right] = 2\ln 2 - \frac{3}{4}$

7. Лице на равнинна фигура

Деф| $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$



Деф| Нека $M_0(x_0,y_0) \in \mathbb{R}^2$, $\delta > 0$

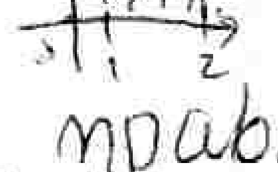
$B_\delta(x_0,y_0) = B_\delta(M_0) = \{M \in \mathbb{R}^2 : |MM_0| < \delta\} = \{(x,y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}$

Деф| Нека $\overline{X} \subset \mathbb{R}^2$. Казваме, че $\overline{X}$ е огр., ако $\exists\ B_r(0,0) : \overline{X} \subset B_r(0,0)$

→ не е огр.

Деф| Правоъг. $P \in \mathbb{R}^2$ е всяко м-во от вида $P = \{(x,y) : a \propto x \propto b,\ c \propto y \propto d\}$, където $\propto \leftrightarrow <, \leq$

Пример: $P = \{(x,y) : 1 \leq x < 2,\ 0 \leq y \leq 3\}$



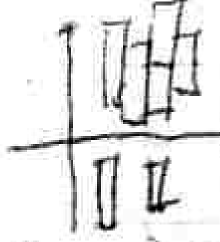
→ не → успоредни на Ox и Oy

Деф| Нека $P = \{(x,y) : a \propto x \propto b,\ c \propto y \propto d\}$ е прав. Вътрешност на правоъгълник P, наричаме м-вото $P^\circ = \{(x,y) : a < x < b,\ c < y < d\}$

Деф| Лице на правоъгълник $P = \{(x,y) : a < x < b,\ c < y < d\}$ наричаме числото $S(P) = (b-a)(d-c)$

Деф| М-вото $K \subset \mathbb{R}^2$ се нарича клетъчно, ако $\exists\ \{P_i\}_{i=1}^n$, P_i – правоъг., $P_j^\circ \cap P_i^\circ = \emptyset$, $\forall i,j = 1 \div n$, $i \neq j$

$K = \bigcup_{i=1}^n P_i$



имат най-мн. общи гр.

Деф| Нека K – кл. м-во. $K = \bigcup_{i=1}^n P_i$: $P_i \cap P_j^\circ = \emptyset$ $\forall i,j = 1 \div n$, $i \neq j$. Лице на K наричаме числото:

$S(K) = \sum_{i=1}^n S(P_i)$

Забележка: 1) Ако $\exists$ кл. м-во $K = \bigcup_{i=1}^{n} P_i = \bigcup_{s=1}^{m} Q_s$

$(P_i^\circ \cap P_j^\circ = \emptyset,\ Q_r^\circ \cap Q_l^\circ = \emptyset,\ r \neq l,\ i \neq j)$

$\Rightarrow \sum_{i=1}^{n} S(P_i) = \sum_{s=1}^{m} S(Q_s) = S(K)$

2) K_1 и K_2 - кл. м-ва : $K_1^\circ \cap K_2^\circ = \emptyset \Rightarrow$

$S(K_1 \cup K_2) = S(K_1) + S(K_2)$

? 3) Ако $K_1 \subset K_2 \Rightarrow S(K_1) \leq S(K_2)$

4) Ако $K \in$ клетъчно м-во и $x_0 \in \mathbb{R}^2 \Rightarrow S(x_0 + K) = S(K)$

$x_0 + K = \{x_0 + x : x \in K\}$

Деф| Нека $G \subset \mathbb{R}^2$. Казваме, че __м-вото $G \in$ измеримо__, ако за всяко $\varepsilon > 0$, $\exists$ кл. м-во A и B:

1) $A \subset G \subset B$

2) $S(B) - S(A) < \varepsilon$

Деф| Нека м-вото $G \subset \mathbb{R}^2$ е измеримо

$\Rightarrow$ едно число $S(G)$ се нарича __лице на G__, ако $\forall$ кл. м-во K: $K \subset G \subset K \Rightarrow S(K) \leq S(G) \leq S(K)$

Т| $\forall$ измеримо м-во $E \subset \mathbb{R}^2$ има единствено лице.

Д-во:

Нека E - изм. м-во в $\mathbb{R}^2$

$\forall$ клетъчно м-во k, K $\quad k \subset E \subset K \Rightarrow S(k) \leq S(K)$

$\Rightarrow \exists$ ~~sup~~ $S(k) \leq \sup_{k \subset E} S(k) \leq S(K) \Rightarrow$

$\exists \inf_{E \subset K} S(K) \Rightarrow S(k) \leq \sup S(k) \leq \inf S(\mathcal{K}) \leq S(\mathcal{K})$ за $\forall k, K: k \subset E \subset K$

$\Rightarrow$ Ако $S(E)$ е лице на $E \Rightarrow S(E) \leq \inf_{E \subset K} S(\mathcal{K})$

$$(+)\begin{cases} \inf_{E \subset \mathcal{K}} S(\mathcal{K}) \leq S(\mathcal{K}) \\ -\sup_{k \subset E} S(k) \leq -S(k) \end{cases}$$

$\inf_{E \subset K} S(K) - \sup_{k \subset E} S(k) \leq S(\mathcal{K}) - S(k) \leq S(\mathcal{K}_\varepsilon) - S(k_\varepsilon) < \varepsilon$

E - измеримо м-во $\Rightarrow \forall \varepsilon > 0 . \exists k, \mathcal{K}$: 1) $k_\varepsilon \subset E \subset \mathcal{K}_\varepsilon$

2) $S(\mathcal{K}_\varepsilon) - S(k_\varepsilon) < \varepsilon$

$\Rightarrow \inf S(\mathcal{K}) - \sup_{k \subset E} S(k)$ ~~$< \varepsilon$~~ $= 0$

$\Rightarrow \underbrace{\inf_{E \subset \mathcal{K}} S(\mathcal{K}) = \sup_{k \subset E} S(k)}_{S(E)} \Rightarrow$ лицето е !

Т (Критерий за измеримост)|: Мн. $G \subset \mathbb{R}^2$ е измеримо $\Leftrightarrow$

$\forall \varepsilon > 0, \exists$ изм. м-ва E, F: 1) $E \subset G \subset F$

2) $S(F) - S(E) < \varepsilon$

Д-во:

Ако G- измеримо $\overset{\text{Деф}}{\Rightarrow} \forall \varepsilon > 0, \exists$ кл. м-ва $k, \mathcal{K}$: 1) $k \subset G \subset \mathcal{K}$

2) $S(\mathcal{K}) - S(k) < \varepsilon$

$\Leftarrow$| Нека $\forall \varepsilon > 0, \exists$ изм. м-ва E и F: 1) $E \subset G \subset F$

2) $S(F) - S(E) < \frac{\varepsilon}{3}$ (1) ?

Т.к. $E \subset G$ е изм, то $\quad \forall \frac{\varepsilon}{3} > 0, \exists$ кл. м-во $k \subset E$: $\quad S(F) - S(k) < \frac{\varepsilon}{3}$ (2)

Т.к. $S(E) = \sup_{k \subset E} S(k)$

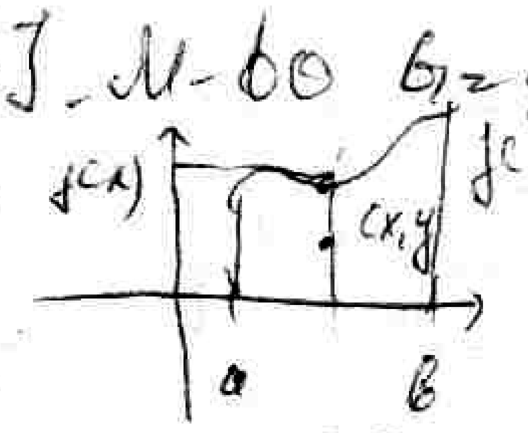
Т.к. $G \subset F$ е изм. м-во $\Rightarrow \forall \frac{\varepsilon}{3} > 0 \ \exists$ кл. во K_ε: $F \subset K_\varepsilon$ и $S(K_\varepsilon) - S(F) < \frac{\varepsilon}{3}$ (3)

т.к. ~~$G \subset F \subset K$~~ $S(F) = \inf\limits_{F\subset K} S(K)$ 1) $K_\varepsilon \subset E \subset G \subset F \subset K_\varepsilon$

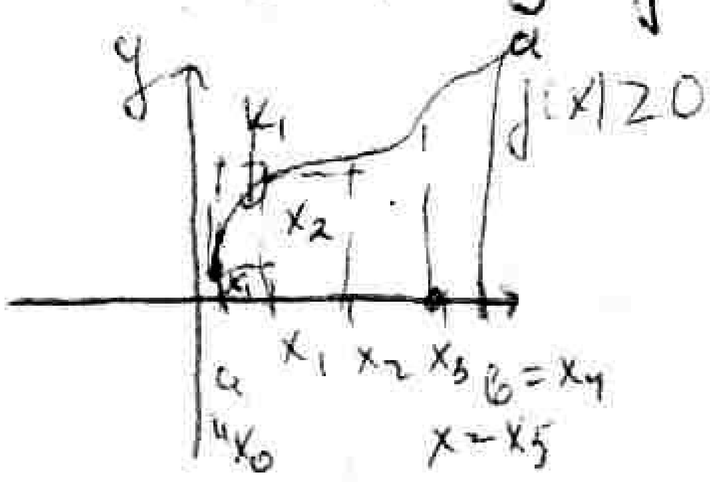
2) $S(K_{\varepsilon'}) - S(k_{\varepsilon'}) = [S(K_{\varepsilon'}) - S(F)] + [S(F) - S(E)] + [S(E) - S(k_{\varepsilon'})] \overset{1,2,3}{<} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

$\overset{\text{Деф}}{\Rightarrow}$ G е измеримо

Деф| Нека ф-та $f(x) \ge 0$ и непр. в/у $[a,b]$. М-во $G = \{(x,y) : a \le x \le b,\ 0 \le y \le f(x)\}$ се нарича криволинеен трапец



Т| Нека $f(x) \ge 0$ и непр. в/у $[a,b] \Rightarrow$ криволин. тр. $G = \{(x,y) : a \le x \le b,\ 0 \le y \le f(x)\}$ е изм. м-во в $\mathbb{R}^2$ и $S(G) = \int_a^b f(x)\,dx$



Д-во:

Нека $\tau = \{x_i\}_{i=0}^n$ е разб. на $[a,b]$

$m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$ $\quad \forall i = 1 \div n$

$k_i = [x_{i-1}, x_i] \times [0, m_i]$

$K_i = [x_{i-1}, x_i] \times [0, M_i]$

Нека $k = \bigcup_{i=1}^n k_i$, $K = \bigcup_{i=1}^n K_i \Rightarrow k, K$ - кл. м-ва, $k \subset G \subset K$

$S(k) = \sum_{i=1}^n S(k_i) = \sum_{i=1}^n m_i \Delta x_i = s_\tau(f)$

$S(K) = \sum_{i=1}^n M_i \Delta x_i = S_\tau$

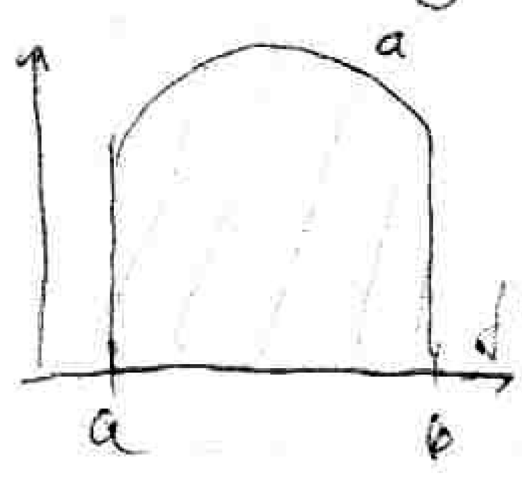
$f(x)$ е непр. в/у $[a,b] \Rightarrow f(x)$ е инт. $\overset{\text{кр. инт.}}{\Rightarrow} \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=0}^n :$
$s_\tau < S_\tau \Rightarrow S_\tau - s_\tau < \varepsilon$

$\Rightarrow \exists$ кл. м-ва k, K: 1) $k \subset G \subset K$
2) $S(K) - S(k) = S_\tau - s_\tau < \varepsilon$

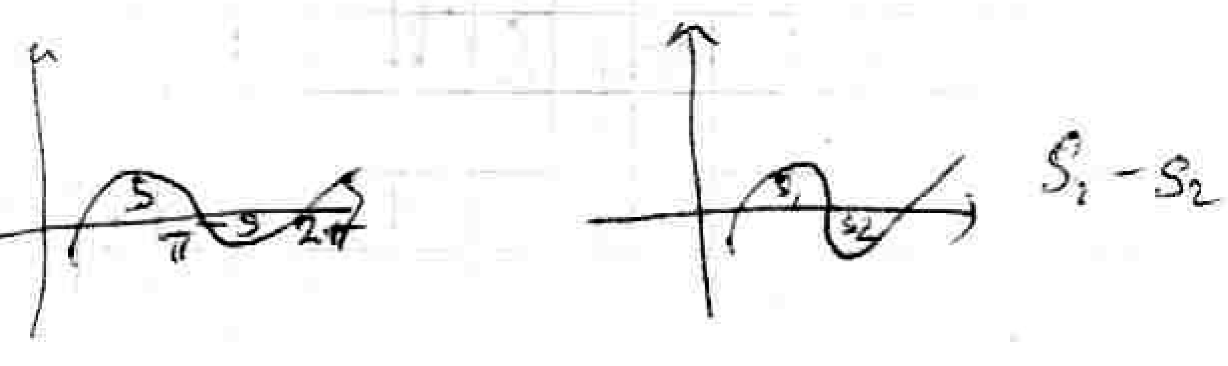
$\Rightarrow G$ е измеримо

$S(G) = \sup\limits_{k \subset G} S(k) = \sup s_\tau = \int_a^b f(x)\,dx$

Деф| $f(x), g(x)$ - непр. в/у $[a,b]$
$D = \{(x,y) : a \le x \le b,\ f(x) \le y \le g(x)\}$, D - изм. м-во
$S(D) = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b [g(x) - f(x)]\,dx$

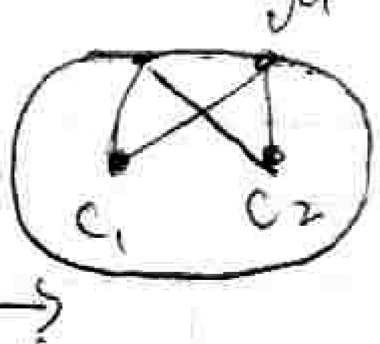


$\int_a^b f(x)\,dx = -S(G)$



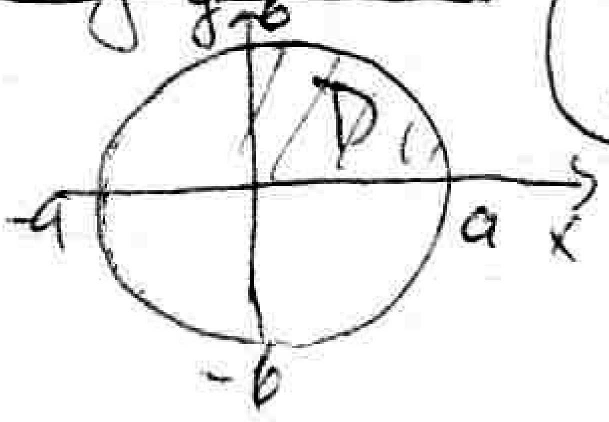
$S_1 - S_2$

Деф| елипса



$\mathcal{E} = \{M : |MC_1| + |MC_2| = const\}$

$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ уравнение на елипса



$\Rightarrow$ Ако $(x,y) \in \mathcal{E} \Rightarrow (-x,y), (x,-y), (-x,-y) \in \mathcal{E}$
$S(\mathcal{E}) = 4S(D)$,

$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$, но $y \ge 0$

$D = \{(x,y) : 0 \le x \le a,\ 0 \le y \le b\sqrt{1 - \frac{x^2}{a^2}}\} \Rightarrow$

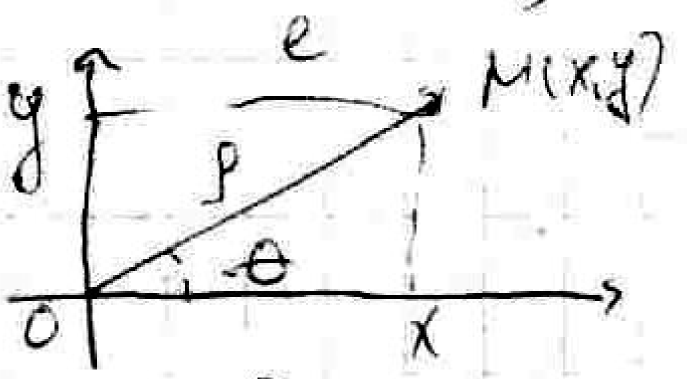
$S(D) = \int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx = ab\int_0 \sqrt{1 - \left(\frac{x}{a}\right)^2}\,d\left(\frac{x}{a}\right) = \quad t = \frac{x}{a}$

$= ab\int_0^1 \underbrace{\sqrt{1-t^2}}_{\pi/4}\,dt = \frac{\pi ab}{4} \Rightarrow S(E) = 4S(D) = \pi ab \Rightarrow$

$$\boxed{S(E) = \pi ab}$$

• Ако $a = b = R \Rightarrow S_{кръг} = \pi R^2$

Деф Полярна координатна с-ма $M(\theta, \rho)$, $\rho(0,0)$ $\Big|$ $0 \le \theta < 2\pi$, $0 < \rho$



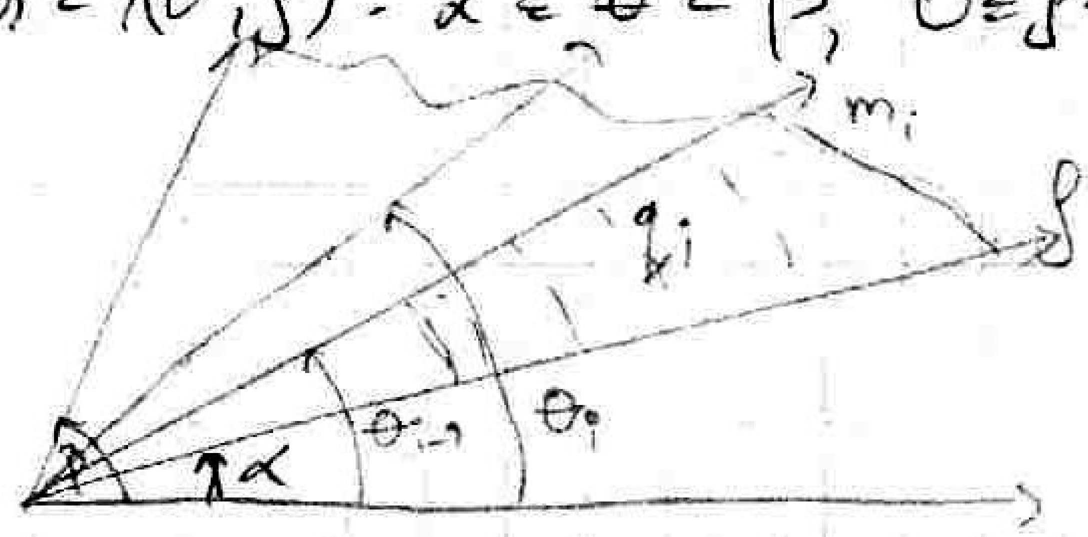
$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ } връзка м/у дек. и пол. к. с-ма

Деф Нека $\rho: \rho(\theta)$ - непр. ф-я $[\alpha, \beta] \subset [0, 2\pi]$

$G = \{(\theta, \rho): \alpha \le \theta \le \beta,\ 0 \le \rho \le \rho(\theta)\}$ $\rightarrow$ криволинеен сектор

Т Нека ф-та $\rho = \rho(\theta)$ е непр в/у $[\alpha, \beta]$. Тогава кр. сектор

$G = \{(\theta, \rho): \alpha \le \theta \le \beta,\ 0 \le \rho \le \rho(\theta)\}$ е измр. м-во и $S(G) = \frac{1}{2}\int_\alpha^\beta \rho^2(\theta)\,d\theta$



Д-во: Нека $\tau = \{\theta_i\}_{i=0}^n$ - разб. на $[\alpha, \beta]$

$\alpha = \theta_0 < \theta_1 < \dots < \theta_{n-1} < \theta_n = \beta$

$[\theta_{i-1}, \theta_i] \rightarrow \Delta\theta_i = \theta_i - \theta_{i-1}$

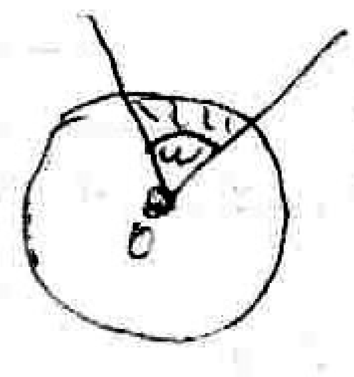
$m_i = \inf\limits_{\theta \in [\theta_{i-1}, \theta_i]} \rho(\theta)$, $M_i = \sup\limits_{\theta \in [\theta_{i-1}, \theta_i]} \rho(\theta)$

$q_i = \{(\theta, \rho): \theta_{i-1} \le \theta \le \theta_i,\ 0 \le \rho \le m_i\}$ $\rightarrow$ кръгов ъглов сектор

$Q_i = \{(\theta, \rho): \theta_{i-1} \le \theta \le \theta_i,\ 0 \le \rho \le M_i\}$ } изм. м-ва

Нека $q = \bigcup_{i=1}^n q_i$ } измерими $q \subset G \subset Q$

$Q = \bigcup_{i=1}^n Q_i$



$\frac{\pi r^2 \omega}{2\pi} = \frac{1}{2} r^2 \omega$

$S(q) = \sum_{i=1}^n S(q_i) = \sum_{i=1}^n \frac{1}{2} m_i^2 \Delta\theta_i$ $\Rightarrow$ $S(q) \rightarrow s_\tau(\frac{1}{2}\rho^2(\theta))$

$S(Q) = \sum_{i=1}^n S(Q_i) = \sum_{i=1}^n \frac{1}{2} M_i^2 \Delta\theta_i$ $\quad$ $S(Q) \rightarrow S_\tau(\frac{1}{2}\rho^2(\theta))$

$\Rightarrow f(\theta) = \frac{1}{2}\rho^2(\theta)$ е непр. в/у $[a, b] \rightarrow$ интегруема в/у $[\alpha, \beta] \Rightarrow$

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0: \forall \tau = \{\theta_i\}_{i=0}^n, \delta_\tau < \delta \Rightarrow$

$$\underbrace{S_\tau(\tfrac{1}{2}\rho^2) - s_\tau(\tfrac{1}{2}\rho^2)}_{\Delta} < \varepsilon, \text{ но}$$

1) $\Delta = S(Q) - S(q) < \varepsilon$
2) $q \subset G \subset Q$
} $\Rightarrow G$ е измеримо м-во

$S(G) = \sup\limits_{q \subset G} S(q) = \sup s_\tau(\frac{1}{2}\rho^2) = \int_\alpha^\beta \frac{1}{2}\rho^2(\theta)\,d\theta = \frac{1}{2}\int_\alpha^\beta \rho^2(\theta)\,d\theta$

Деф| Окр.(R): $\rho = \rho(\theta) = R, \quad 0 \le \theta \le 2\pi$

$B_R(0): \{(\theta, \rho): 0 \le \theta \le 2\pi, \ 0 \le \rho \le R\}$

$$S(B_R(0)) = \frac{1}{2}\int_0^{2\pi} R^2\, d\theta = \frac{1}{2}R^2 \cdot \int_0^{2\pi} 1\, d\theta = R^2 2\pi \cdot \frac{1}{2} = \pi R^2$$

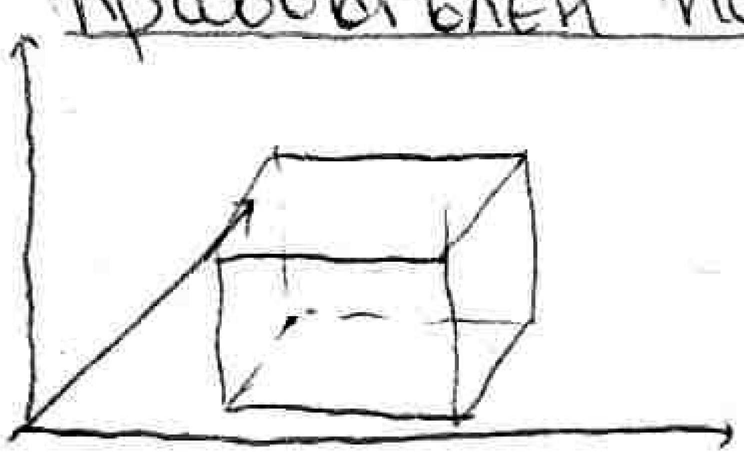
- $\rho = a(1+\cos\theta), \quad 0 \le \theta \le 2\pi$

$$S(G) = \frac{1}{2}\int_0^{2\pi} a^2(1+\cos\theta)^2\, d\theta = \frac{a^2}{2}\int_0^{2\pi}(1+\cos\theta+\cos^2\theta)\, d\theta =$$
$$= \frac{a^2}{2}\left[\int_0^{2\pi} 1\, d\theta + 2\int_0^{2\pi}\cos\theta\, d\theta + \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\, d\theta\right] =$$
$$= \frac{a^2}{2}\left[2\pi + 2\sin\theta\Big|_0^{2\pi} + \frac{1}{2}\left(2\pi + \frac{1}{2}\sin 2\theta\Big|_0^{2\pi}\right)\right] =$$
$$= \frac{a^2}{2}(2\pi + \pi) = \frac{3}{2}\pi a^2$$

8. Обем на тяло с известно напречно сечение. Обем на ротационно тяло

Деф| $\Pi = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle \times \langle a_3, b_3 \rangle$, където $a_i, b_i \in \mathbb{R}$, $i = 1,2,3$, $\langle \in \{<, \le\}$, $\rangle \in \{>, \ge\}$

правоъгълен паралелепипед



Вътрешност на Π: $\Pi^0 = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$

Деф| Клетъчно тяло наричаме $\forall$ мн. $K = \bigcup_{i=1}^{n} \Pi_i$: $\Pi_i^0 \cap \Pi_j^0 = \emptyset \ (\forall i, j = 1 \div n, \ i \ne j)$

Деф| $V(\Pi) = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$ – обем на Π

$V(K) = \sum_{i=1}^{n} V(\Pi_i)$ – обем на кл. тяло

Деф| Нека $\Omega \subset \mathbb{R}^3$. Казваме, че Ω е <u>измеримо м-во/тяло</u>, ако за $\forall \varepsilon > 0$, $\exists$ кл. тяло k, K: 1) $k \subset \Omega \subset K$
2) $V(K) - V(k) < \varepsilon$

Деф| Нека Ω е измеримо тяло. <u>Обем на Ω</u> се нарича такова число $V(\Omega)$: $\forall k, K$ (кл. тела): $k \subset \Omega \subset K \Rightarrow$
$V(k) \le V(\Omega) \le V(K)$

Т| Ако Ω е измеримо тяло в $\mathbb{R}^3$, то $\exists!\ V(\Omega)$ на Ω. При това

$$V(\Omega) = \sup_{k \subset \Omega} V(k) = \inf_{\Omega \subset K} V(K)$$

Д-во:

Нека Ω – измеримо м-во в $\mathbb{R}^3$

$\forall$ клетъчно м-во k, K: $k \subset \Omega \subset K \Rightarrow V(k) \le V(K)$

$\Rightarrow \exists\ V(k) \le \sup_{k \subset \Omega} V(k) \le V(K)$

$\Rightarrow \exists \inf_{\Omega \subset K} V(K) \Rightarrow V(k) \le \sup V(k) \le \inf V(K) \le V(K)$ за $\forall k, K \quad k \subset \Omega \subset K$

$\Rightarrow$ Ако $V(\Omega)$ е обем на $\Omega \Rightarrow V(\Omega) \le \inf_{\Omega \subset K} V(K)$

$$+\begin{cases} \inf_{\Omega \subset K} V(K) \le V(K) \\ -\sup V(k) \le -V(k) \end{cases}$$