Д-во:

Нека $f(x)$ – м.р. в/у $[a, b]$

τ – разб., $\tau = \{x_i\}_{i=0}^{n}$, р. на $[a, b]$

$\forall\, x \in [x_{i-1}, x_i]$

$$\Rightarrow f(x_{i-1}) \le f(x) \le f(x_i)$$

$$m_i = \inf_{x\in[x_{i-1}, x_i]} f(x) = f(x_{i-1})$$

$$M_i = \sup_{x\in[x_{i-1}, x_i]} f(x) = f(x_i)$$

$$S_\tau - s_\tau = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x_i$$

$\forall\, \varepsilon > 0,\ \delta = \delta(\varepsilon) = \dfrac{\varepsilon}{2[f(b)-f(a)]} > 0.$

$\forall\, \tau = \{x_i\}_{i=0}^{n}: \delta_\tau < \delta = \dfrac{\varepsilon}{2(f(b)-f(a))}$

$$\Rightarrow S_\tau - s_\tau = \sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]\Delta x_i < \sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]\,\delta = \delta\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]$$

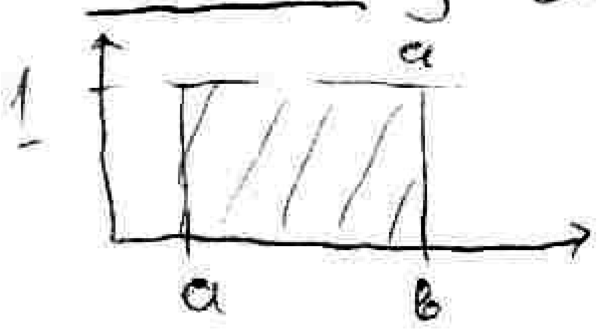
$$= \frac{\varepsilon}{2[f(b)-f(a)]}\cdot f(b)-f(a) = \frac{\varepsilon}{2} < \varepsilon.$$

$\Rightarrow$ (кр. за инт.) $f(x)$ е инт. в/у $[a, b]$

Th| $f(x)$ – огр. в/у $[a, b]$ и има кр. бр. т. на прекъсване, тогава $f(x)$ – инт. в/у $[a, b]$

4 Свойства на определения интеграл

Св. 1| $\int_a^b 1\, dx = b - a$



Д-во:

$$\int_a^b 1\, dx = \sup_\tau s_\tau \ *$$

Нека $\tau = \{x_i\}_{i=0}^{n}$ – разбиване на $[a, b]$

$\forall\, i = 1 \div n: m_i = \inf_{x\in[x_{i-1}, x_i]} f(x) = \inf_{x\in[x_{i-1}, x_i]} 1 = 1$

$$\Rightarrow s_\tau = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 1\,\Delta x_i = b - a$$

$$* \Rightarrow \sup_\tau (b-a) = b - a = \int_a^b 1\, dx$$

Оп| $\int_a^a f(x)\, dx = 0$

Оп| Ако $a < b$, то $\int_b^a f(x)\, dx := -\int_a^b f(x)\, dx$

Св 2| Нека $f(x)$ и $g(x)$ – интегр. в/у $[a, b]$, $\lambda \in \mathbb{R}$

$\Rightarrow f(x) + g(x)$, $\lambda f(x)$ са интегр. в/у $[a, b]$

1) $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

2) $\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx$

Следствие 1: При усл. на св. 2 $f(x)-g(x)$ е инт. в/у $[a,b]$ и
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Д-во:

1) Нека $\tau=\{x_i\}_{i=0}^n$ – разб. на $[a,b]$, $\xi=\{\xi_i\}_{i=1}^n$, $\xi_i\in[x_{i-1},x_i]$ $(\forall i=1\div n)$

$$\sigma_\tau(f+g;\xi)=\sum_{i=1}^n (f+g)(\xi_i)\Delta x_i=\sum_{i=1}^n [f(\xi_i)+g(\xi_i)]\Delta x_i=$$
$$=\sum_{i=1}^n f(\xi_i)\Delta x_i+\sum_{i=1}^n g(\xi_i)\Delta x_i=\sigma_\tau(f,\xi)+\sigma_\tau(g;\xi)$$

Т.к. $f(x)$ и $g(x)$ – инт. в/у $[a,b] \Rightarrow \exists I_1, I_2 : \forall \varepsilon>0, \exists \delta=\delta(\varepsilon)>0 :$ $\forall \tau=\{x_i\}_{i=1}^n$, $\delta_\tau<\delta$, $\forall \xi=\{\xi_i\}_{i=1}^n$ $\Big]\star$

$|I_1-\sigma_\tau(f,\xi)|<\frac{\varepsilon}{2}$ (1)

$|I_2-\sigma_\tau(g,\xi)|<\frac{\varepsilon}{2}$

Разгл. $|\sigma_\tau(f+g,\xi)-(I_1+I_2)|=|\sigma_\tau(f,\xi)+\sigma_\tau(g,\xi)-(I_1+I_2)|\le$
$$|\sigma_\tau(f,\xi)-I_1|+|\sigma_\tau(g;\xi)-I_2| \overset{(1)}{<} \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

$\Rightarrow$ за $f+g$, $\exists I_1+I_2 : \star \Rightarrow (2)$, т.е. $(f+g)(x)$ е инт в/у $[a,b]$ и
$$\int_a^b [f(x)+g(x)]\,dx=I_1+I_2=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$$

2) $\exists I_1 : \forall \varepsilon>0, \exists \delta=\delta(\varepsilon)>0 : \forall \tau=\{x_i\}_0^n, \forall \xi \Rightarrow$
$$|\sigma_\tau(f,\xi)-I_1|<\frac{\varepsilon}{|\lambda|}\ (3)\quad (\lambda\neq 0)\ (\text{за } \lambda=0 \text{ е изп.})$$
$$\sigma_\tau(\lambda f,\xi)=\lambda\,\sigma_\tau(f,\xi)$$
$$\Rightarrow |\sigma(\lambda f,\xi)-\lambda I_1|=|\lambda\,\sigma_\tau(f,\xi)-\lambda I_1|=|\lambda|\,|\sigma_\tau(f,\xi)-I_1| \overset{(3)}{<} \cancel{|\lambda|}\frac{\varepsilon}{\cancel{|\lambda|}}=\varepsilon$$

$\Rightarrow \lambda f$ е инт. в/у $[a,b]$ и $\int_a^b \lambda f(x)\,dx=\lambda\int_a^b f(x)\,dx$

$\overset{1,2}{\Longrightarrow}$ ~~f(x)~~ $R[a,b]=\{f(x)$: инт. по Риман в/у $[a,b]\}$ е л.п.

$\int_a^b f(x)\,dx : R[a,b]\to R$ – л. оп., линеен функционал

Св. 3| Ако $f(x)$ е инт. в/у $[a,b]$ и неотр. в/у $[a,b] \Rightarrow \int_a^b f(x)\,dx\ge 0$

Д-во:

$\int_a^b f(x)\,dx=\sup_\tau s_\tau$ $(\star)$

$s_\tau=\sum_{i=1}^n m_i\,\Delta x_i$ и $m_i=\inf_{x\in[x_{i-1},x_i]} f(x)\ge 0$, $(\forall i=1\div n)$

$\Rightarrow s_\tau\ge \cancel{\sum_{i=1}^n m_i \Delta x_i} 0 \Rightarrow \sup_\tau s_\tau\ge 0 \overset{\star}{\Rightarrow} \int_a^b f(x)\,dx\ge 0$

Следствие 2| Ако $f(x)$ и $g(x)$ – инт. и $f(x)\ge g(x)$ в/у $[a,b]\Rightarrow$
$$\int_a^b f(x)\,dx\ge\int_a^b g(x)\,dx$$

Д-во:

т.к. $f(x)-g(x)$ в/у $[a,b] \Rightarrow f(x)-g(x)\ge 0$ в/у $[a,b] \overset{\text{св. 3}}{\Rightarrow}$
$$\int_a^b [f(x)-g(x)]\,dx\ge 0 \underset{\text{сл. 1}}{\Longleftrightarrow} \int_a^b f(x)\,dx-\int_a^b g(x)\,dx \Rightarrow \int_a^b f(x)\,dx\ge\int_a^b g(x)\,dx$$

Св. 4] Ако $f(x)$ – инт. в/у $[a,b] \Rightarrow |f(x)|$ инт в/у $[a,b]$ и
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Д-во:

$|f(x)| \geq f(x)$ и $|f(x)| \geq -f(x)$ $\xRightarrow{\text{сл. 1}}$

$$\int_a^b |f(x)|\,dx \geq \int_a^b f(x)\,dx \text{ и } \int_a^b |f(x)|\,dx \geq -\int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b |f(x)|\,dx \geq \left|\int_a^b f(x)\,dx\right|$$

Св. 5] Ако $f(x)$ е инт. в/у $[a,b] \Rightarrow f(x)$ е инт. в/у всеки подинт. $[c,d] \subset [a,b]$

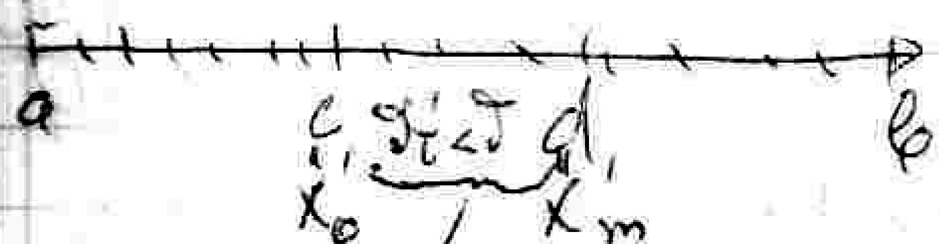
Д-во:

$f(x)$ е инт. в/у $[a,b] \Rightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0,$

$\forall \tau = \{x_i\}_{i=0}^n, \delta_\tau < \delta \Rightarrow \sum_{i=1}^n \omega_i(f).\Delta x_i < \varepsilon$

Нека има произв. разбиване:

$\forall \tau' = \{x_i\}_{i=0}^m$ на инт. $[c,d]$: $\delta_{\tau'} < \delta$



$\Rightarrow \exists \tau = \{x_i\}_{i=0}^n \supset \tau'$: $\delta_\tau < \delta$

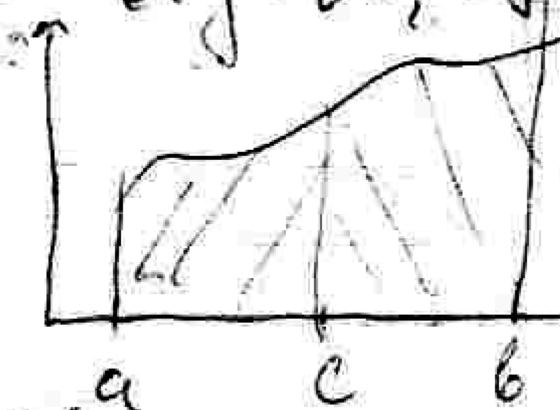
$$\sum_{i=0}^m \omega_i(f)\,\Delta x_i' \leq \sum_{i=1}^n \omega_i(f)\,\Delta x_i < \varepsilon$$

$\Delta x_i' = x_i'' - x_{i-1}'$ $(\forall i = 1 \div n)$

$\xRightarrow{\text{кр. за инт.}}$ $f(x)$ е също инт. в/у $[c,d]$

Свойство 6:] Ако $f(x)$ е инт. в/у $[a,b]$ и $c \in [a,b] \Rightarrow$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



$\rightarrow$ за лицата

Д-во:

Нека $c \in [a,b]$ $a < c < b$

$\xRightarrow{\text{св. 5}}$ $f(x)$ е инт. в/у $[a,c]$ и $[c,b]$

Нека $I = \int_a^b f(x)\,dx$, $I_1 = \int_a^c f(x)\,dx$, $I_2 = \int_c^b f(x)\,dx$.

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$: 1) $\forall \tau^{[a,b]} = \{x_i\}_{i=0}^n$: $\delta_{\tau^{[a,b]}} < \delta \Rightarrow$

$\forall \xi^{[a,b]} = \{\xi\}_{i=1}^n \Rightarrow |I - \sigma_{\tau^{[a,b]}}(f, \xi^{[a,b]})| < \varepsilon$

2) $\forall \tau^{[a,c]} = \{x_i'\}_{i=0}^m$: $\delta_{\tau^{[a,c]}} < \delta$

$\forall \xi^{[a,c]} \Rightarrow |I_1 - \sigma_{\tau^{[a,b]}}(f, \xi^{[a,b]})| < \varepsilon$

3) $\forall \tau^{[c,b]} = \{x_i''\}_{i=0}^k$: $\delta_{\tau^{[a,b]}} < \delta$ $\forall \xi^{[a,b]} \Rightarrow$

$\forall \xi^{[c,b]} \Rightarrow |I_2 - \sigma_{\tau^{[c,b]}}(f, \xi^{[c,b]})| < \varepsilon$

Нека $\tau^{[a,c]}: \delta_\tau[a,c] < \delta$ и $\tau^{[c,b]}: \delta_\tau[c,b] < \delta$ $\Rightarrow$ $\tau^{[a,b]} = \tau^{[a,c]} \cup \tau^{[c,b]}$, $\delta_{\tau^{[a,b]}} < \delta$ (*),

$\xi_0^{[a,c]}, \xi_0^{[c,b]}, \xi_0^{[a,b]} = \xi_0^{[a,c]} \cup \xi_0^{[c,b]}$ – фиксирани са

$\Rightarrow$ за (*) иде са в сила с-вата за I от 1, 2, 3

$$\Rightarrow |I - (I_1 + I_2)| = |I - \sigma_{\tau^{[a,b]}}(f; \xi_0^{[a,b]}) + [\sigma_{\tau^{[a,b]}}(f; \xi_0^{[a,b]}) - (I_1 + I_2)]|$$

$$\leq |I - \sigma_{\tau^{[a,b]}}(f, \xi_0^{[a,b]})| + |\sigma_{\tau^{[a,c]}}(f, \xi_0^{[a,c]}) + \sigma_{\tau^{[c,b]}}(f, \xi_0^{[c,b]}) - I_1 - I_2|$$

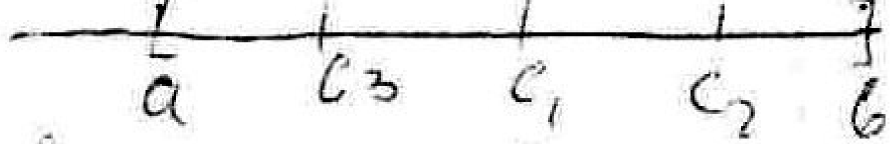
$$\leq |I - \sigma_{\tau^{[a,b]}}(f, \xi_0^{[a,b]})| + |I_2 - \sigma_{\tau^{[c,b]}}(f, \xi_0^{[c,b]})| + |I_1 - \sigma_{\tau^{[a,c]}}(f, \xi_0^{[a,c]})|$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

т.е. $\forall\, \varepsilon > 0 \Rightarrow |I - (I_1 + I_2)| < 3\varepsilon \Rightarrow I - (I_1 + I_2) = 0 \Leftrightarrow I = I_1 + I_2 \Leftrightarrow$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>Следствие 4:</u>



$$\int_{c_1}^{c_3} f(x)\,dx \overset{?}{=} \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx$$

$\int_{c_3}^{c_2} f(x)\,dx = \int_{c_3}^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx$ (подредба) ; $\int_{c_3}^{c_1} f(x)\,dx = \int_{c_1}^{c_2} f(x)\,dx - \int_{c_3}^{c_2} f(x)\,dx \Rightarrow$

$$\int_{c_1}^{c_3} f(x)\,dx = \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx \,!$$

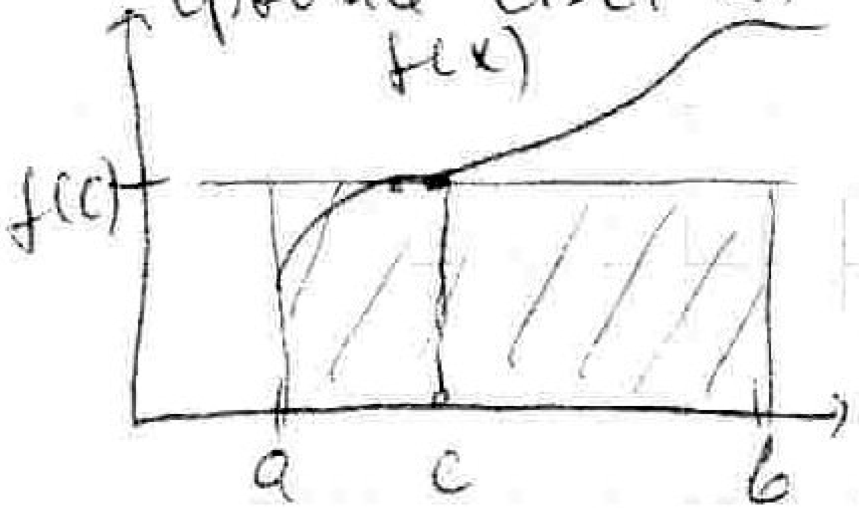
Аналог. за Ор. 5

<u>Свойство 7:</u> Нека $f(x)$ непр. в/у $[a,b] \Rightarrow \exists\, c \in [a,b]$:

$$\boxed{\int_a^b f(x)\,dx = f(c)\cdot(b-a)}$$

(теорема за средната стойност)

• $f(c) = \frac{1}{b-a}\cdot\int_a^b f(x)\,dx$ средна ст-ст на $f(x)$



<u>Д-во</u>:

Т.к. $f(x)$ е непр. в/у $[a,b] \overset{\text{Т.В.}}{\Rightarrow} \exists\, x_0, x_1 \in [a,b]$:

$\forall\, x \in [a,b]$: $f(x_0) \le f(x) \le f(x_1) \Rightarrow$

$$\int_a^b f(x_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(x_1)\,dx \Rightarrow$$

$$f(x_0)(b-a) \le \int_a^b f(x)\,dx \le f(x_1)(b-a) \,:\, b-a \ne 0 \Rightarrow$$

$$f(x_0) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(x_1) \overset{\text{Т.М.С.}}{\Rightarrow} \exists\, c \in [a,b] \Rightarrow f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx, \text{ т.е.:}$$

$$\int_a^b f(x)\,dx = f(c)(b-a)$$