1) Нека $x_{min} \in [a,b]: f(x_{min}) = \inf\limits_{x\in[a,b]} f(x)$ (*)

Нека $x_{max} \in [a,b]: f(x_{max}) = \sup\limits_{x\in[a,b]} f(x)$ (**)

$\Rightarrow \begin{cases} f(x_{min}) \le f(x') \le f(x_{max}) & \forall x' \in [a,b] \\ -f(x_{min}) \ge -f(x'') \ge -f(x_{max}) & \forall x'' \in [a,b] \end{cases} \Bigg| +$

$\Rightarrow |f(x') - f(x'')| \le f(x_{max}) - f(x_{min}) = \sup\limits_{x\in[a,b]} f(x) - \inf\limits_{x\in[a,b]} f(x) = \omega(f)$

2) $\stackrel{*}{\Rightarrow} \forall \frac{\varepsilon}{2} > 0,\ \exists \bar{x}' \in [a,b]: f(x_{min}) \le f(\bar{x}') < f(x_{min}) + \frac{\varepsilon}{2}$ (1)

$\stackrel{**}{\Rightarrow} \forall \frac{\varepsilon}{2} > 0,\ \exists \bar{x}'' \in [a,b]: f(x_{max}) \ge f(\bar{x}'') > f(x_{max}) - \frac{\varepsilon}{2}$ (2)

$\stackrel{1-2}{\Rightarrow} f(\bar{x}') - f(\bar{x}'') < f(x_{min}) - f(x_{max}) + \varepsilon$ (3)

$\stackrel{2-1}{\Rightarrow} f(\bar{x}'') - f(\bar{x}') > f(x_{max}) - f(x_{min}) - \varepsilon$ (4)

$\stackrel{3\&4}{\Rightarrow} |f(\bar{x}') - f(\bar{x}'')| > f(x_{max}) - f(x_{min}) - \varepsilon = \omega(f) - \varepsilon$

Т - критерий за интегруемост:

Нека $f(x)$ - огр. в/у $[a,b]$

ф-ята $f(x)$ - инт. в/у $[a,b] \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0:$
$\forall \tau = \{x_i\}_{i=0}^{n},\ \delta\tau < \delta \Rightarrow \boxed{\sum\limits_{i=1}^{n} \omega_i \Delta x_i < \varepsilon}$

3. Класове интегруеми функции

Т | Нека $f(x)$ - непр. в/у $[a,b]$. Тогава $f(x)$ - инт. в/у $[a,b]$

Д-во:

Т.к. $f(x)$ е непр. в/у $[a,b] \Rightarrow$ равном. непр. в/у $[a,b] \Rightarrow$

$\forall \varepsilon > 0,\ \varepsilon' = \frac{\varepsilon}{2(b-a)} > 0$, т.к. $f(x)$ - равном. н. в/у $[a,b]$, то $\exists \delta = \delta(\varepsilon) > 0, \forall x', x'' \in [a,b]$, $|x' - x''| < \delta \Rightarrow |f(x') - f(x'')| < \varepsilon' = \frac{\varepsilon}{2(b-a)}$

$\forall \tau = \{x_i\}_{i=1}^{n} : \delta\tau < \delta$

$S_\tau - s_\tau = \sum\limits_{i=1}^{n} \omega_i(f)\, \Delta x_i \le \sum\limits_{i=1}^{n} \frac{\varepsilon}{2(b-a)} \Delta x_i = \frac{\varepsilon}{2(b-a)} \cdot \sum\limits_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{2\cancel{(b-a)}} \cancel{(b-a)} < \varepsilon$

$\omega_i(f) = \sup\limits_{x', x'' \in [x_{i-1}, x_i]} |f(x') - f(x'')| \le \sup\limits_{x', x'' \in [x_{i-1}, x_i]} \varepsilon' = \varepsilon' = \frac{\varepsilon}{2(b-a)}$

$|x' - x''| \le |[x_{i-1}, x_i]| \le \delta\tau \le \delta$

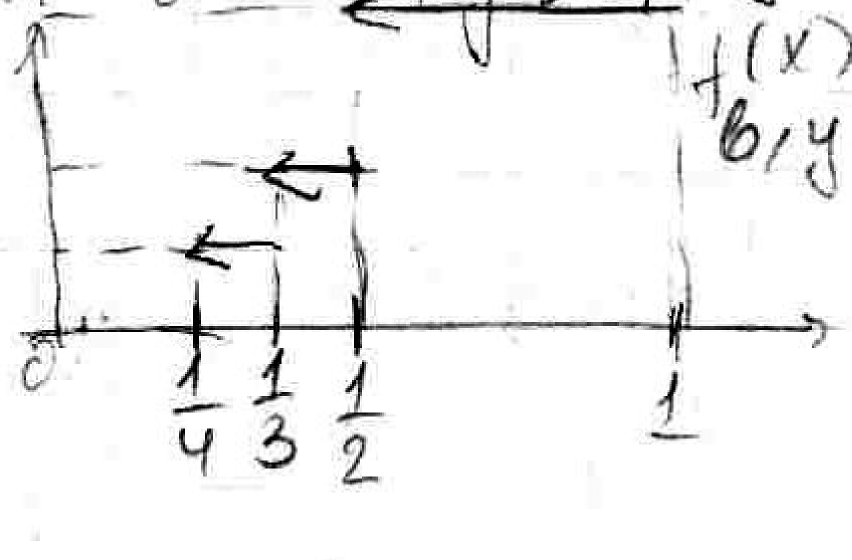
$S_\tau - s_\tau < \varepsilon \Rightarrow$ (кр. за инт.) $f(x)$ е инт. в/у $[a,b]$

Т | Нека $f(x)$ - монотонна в/у $[a,b] \Rightarrow f(x)$ - инт. в/у $[a,b]$

- Постр. $f(x) = \begin{cases} \frac{1}{n}, & x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N} \\ 0, & x = 0 \end{cases}$

$D(f) = [0,1]$

?• $\int\limits_0^1 f(x) =$



$f(x)$ - м.р. в/у $[0,1]$

Д-во:

Нека $f(x)$ – н.р. в/у $[a, b]$
τ – разб., $\tau = \{x_i\}_{i=0}^{n}$, р. на $[a, b]$
$\forall x \in [x_{i-1}, x_i]$

$$\Rightarrow f(x_{i-1}) \leq f(x) \leq f(x_i)$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = f(x_{i-1})$$

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_i)$$

$$S_\tau - s_\tau = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x_i$$

$\forall \varepsilon > 0,\ \delta = \delta(\varepsilon) = \dfrac{\varepsilon}{2[f(b) - f(a)]} > 0.$

$\forall \tau = \{x_i\}_{i=0}^{n} : \delta_\tau < \delta = \dfrac{\varepsilon}{2(f(b) - f(a))}$

$$\Rightarrow S_\tau - s_\tau = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x_i < \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\delta = \delta \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

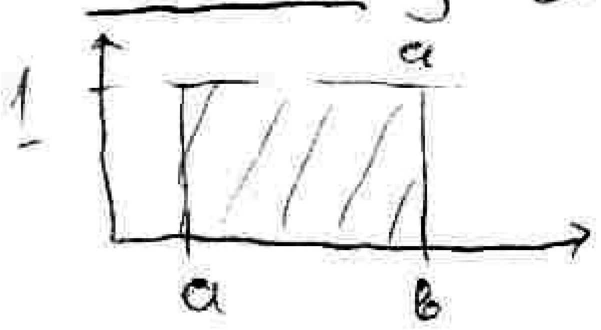
$$= \frac{\varepsilon}{2[f(b) - f(a)]} \cdot f(b) - f(a) = \frac{\varepsilon}{2} < \varepsilon.$$

$\Rightarrow$ (кр. за инт.) $f(x)$ е инт. в/у $[a, b]$

Т° | $f(x)$ – огр. в/у $[a, b]$ и има кр. бр. т. на прекъсване, тогава $f(x)$ – инт. в/у $[a, b]$

4 Свойства на определения интеграл

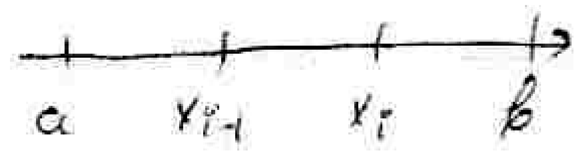
Св. 1 | $\int_a^b 1\,dx = b - a$



Д-во:

$$\int_a^b 1\,dx = \sup_\tau s_\tau \ *$$

Нека $\tau = \{x_i\}_{i=0}^{n}$ – разбиване на $[a, b]$



$\forall i = 1 \div n : m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = \inf_{x \in [x_{i-1}, x_i]} 1 = 1$

$$\Rightarrow s_\tau = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 1\,\Delta x_i = b - a$$

$$* \Rightarrow \sup_\tau (b - a) = b - a = \int_a^b 1\,dx$$

Оп | $\int_a^a f(x)\,dx = 0$

Оп | Ако $a < b$, то $\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$

Св 2 | Нека $f(x)$ и $g(x)$ – интегр. в/у $[a, b]$, $\lambda \in \mathbb{R}$

$\Rightarrow f(x) + g(x)$, $\lambda f(x)$ са интегр. в/у $[a, b]$

1) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

2) $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$