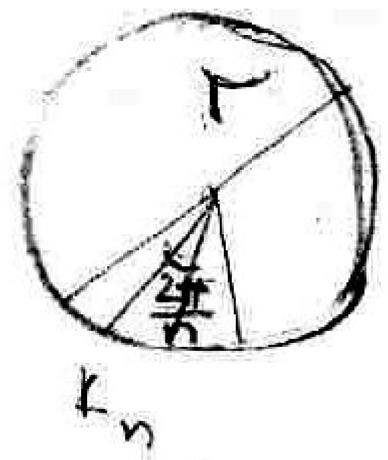
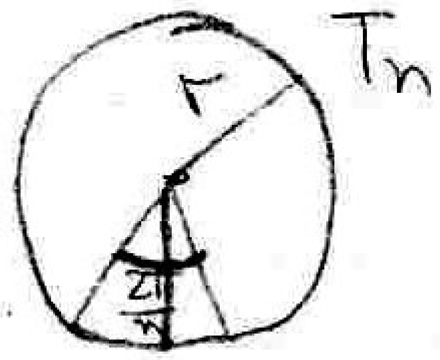
# Анализ II




$$S(K_n)=\frac{n\cdot r^2\cdot\sin\left(\frac{2\pi}{n}\right)}{2}=\pi r^2\overbrace{\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}}^{\to 1}\xrightarrow[n\to\infty]{}\pi r^2$$

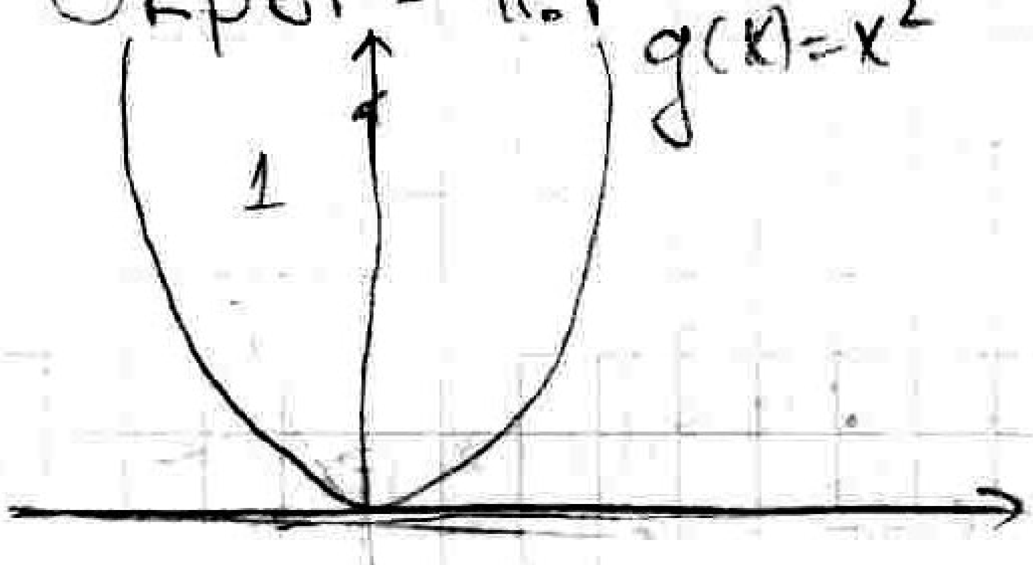



$$\frac{\frac{a}{2}}{r}=tg\frac{\pi}{n}\qquad a=2r\,tg\frac{\pi}{n}$$

$$S(T_n)=\cancel{\sum_{i=1}}\ n\cdot\frac{2r^2\cdot tg\frac{\pi}{n}}{\cancel{2}}=n\cdot r^2\cdot tg\frac{\pi}{n}=$$

$$=\pi r^2\left(\frac{tg\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}\right)\xrightarrow[n\to\infty]{}\pi\cdot r^2\cdot 1=\pi r^2$$

$$S_{кръг}=\pi\cdot r^2$$




$$S(T_n)=\sum_{i=1}^{n-1}\frac{1}{n}\left(\frac{i}{n}\right)^2=\frac{1}{n^3}\sum_{i=1}^{n-1}i^2=$$

$$=\frac{1}{n^{\cancel{3}2}}\cdot\frac{(n-1)\cdot\cancel{n}(2n-1)}{6}=\frac{(n-1)(2n-1)}{6n^2}$$

$$x_i=\frac{i}{n},\ i=0\div n$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$S(T_n)=\frac{(n-1)(2n-1)}{6n^2}=\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\xrightarrow[n\to\infty]{}\frac{1}{6}\cdot 1\cdot 2=\frac{1}{3}$$

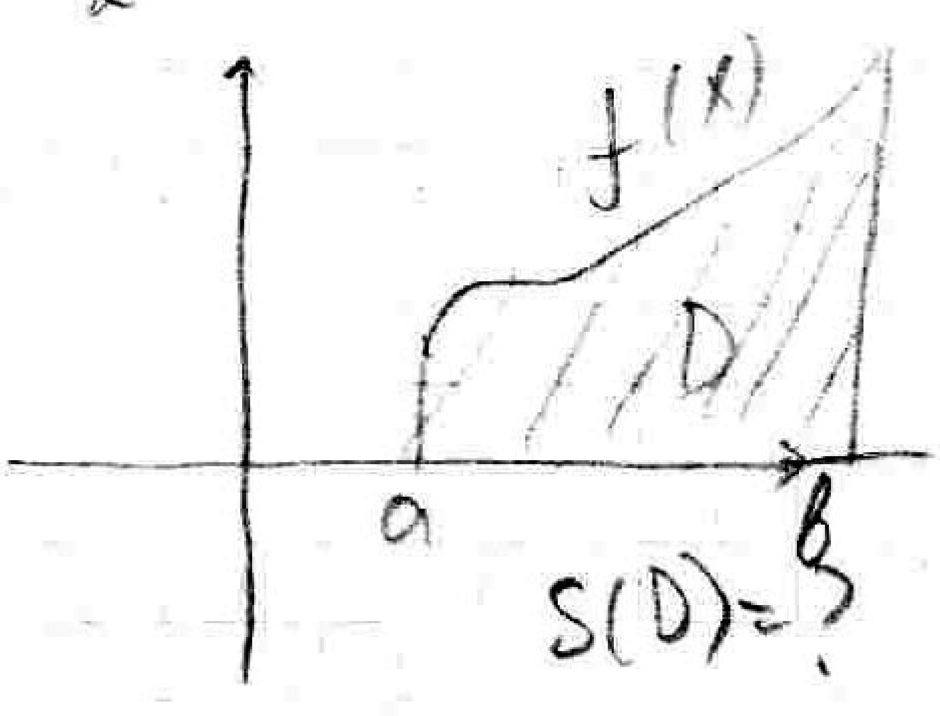
$$S(K_n)=\sum_{i=1}^{n}\frac{1}{n}\cdot\left(\frac{i}{n}\right)^2=\frac{1}{n^3}\sum_{i=1}^{n}i^2=\frac{1}{n^3}\cdot\frac{n\cdot(n+1)(2n+1)}{6}=\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\xrightarrow[n\to\infty]{}$$

$$\xrightarrow[n\to\infty]{}\frac{1}{6}\cdot 1\cdot 2=\frac{1}{3}$$

$$D=\{(x,y):0\le x\le 1,\ 0\le y\le x^2\}$$

$$S(D)=\frac{1}{3}$$

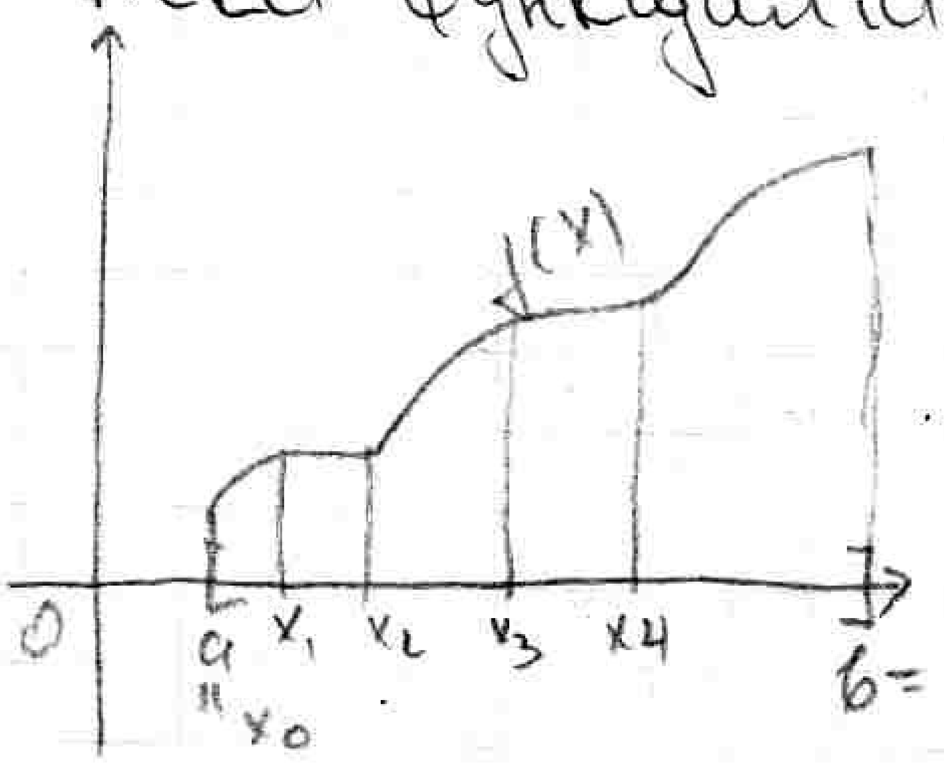
$$S_\Delta=\frac{1}{2}\qquad S_\Delta=S_0-S(D)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$




$S(D)=?$

# 1. Определен интеграл на Риман - определение. Необходимо условие за интегруемост.

Нека функцията $f(x)$ е деф. в/у $[a,b]$




**Деф** $\tau = \{x_i\}_{i=0}^{n}$

$a = x_0 < x_1 < x_2 < x_3 < x_4 < \dots < x_{n-1} < x_n = b$

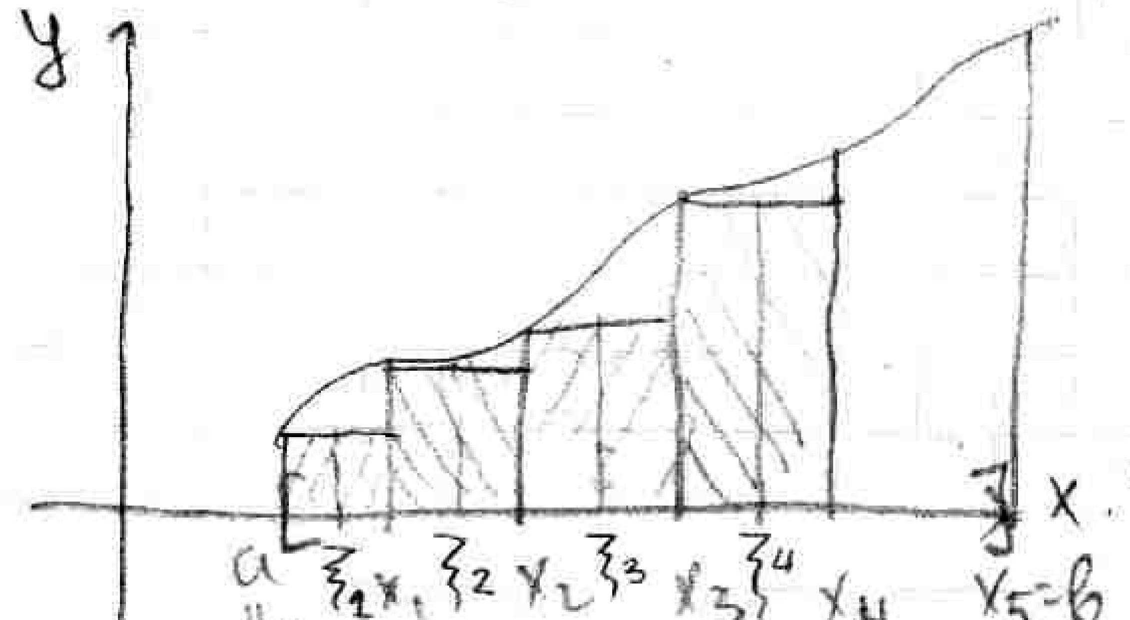
се нарича <u>разбиване на инт. $[a,b]$</u>

~~$\{[x_{i-1}, x_i]\}_{i=1}^{n} : [x_{i-1}, x_i] \cap [x_{i-1}, x_i] = \begin{cases} \emptyset \\ \text{състои се от 1 т.} \end{cases}$~~

$\Delta x_i = x_i - x_{i-1}$, $(\forall\ i = 1 \div n)$,

~~$\forall\ i = 1 \div n : \xi_i \in [x_{i-1}, x_i]$~~

~~$\sum_{i=1}^{n} f(\xi_i)\Delta x_i : \sigma_\tau(f,\xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ сума на Риман~~




$f(x)$ - отг. на разбиването $\tau$ и набора от точки $\xi$, $f(x) \geq 0$, в/у $[a,b]$

$D = \{(x,y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}$ $\sigma_\tau(f,\xi) \approx S_D$

**Деф** $\delta_\tau = \max_{i=1}^{n} \Delta x_i$ - <u>големина на разбиването $\tau$</u>

(*равна на най-големия инт. от разбиването)

**Деф** Дадено е разбиване $\tau$, $\tau = \{x_i\}_{i=0}^{n}$ $a = x_0 < \dots < x_n = b$. Тогава

$\xi = \{\xi_i\}_{i=1}^{n}$, $\xi_i \in [x_{i-1}, x_i]$ $\forall i \in \{1 \dots n\}$.

се <u>нарича набор от точки</u>, отговарящ на разб. $\tau$.

**Деф** $f(x)$ - дефинирана и огр. в/у $[a,b]$, $f(x) \geq 0$ в/у $[a,b]$!

$\tau = \{x_i\}_{i=0}^{n}$, $\xi = \{\xi_i\}_{i=1}^{n} : \xi_i \in [x_{i-1}, x_i]$

$$\boxed{\sigma_\tau(f,\xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i}$$

се нарича <u>сума на Риман</u> за разбиване $\tau$ на ф-я $f$ и стойности $\xi$

* сума на Риман - апроксимация за лицето на фиг. под графиката на $f$

**Деф** Казваме, че ф-та $f(x)$ е интегруема в смисъла на Риман в/у $[a,b]$, ако $\exists I \in \mathbb{R} : \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=1}^{n}, \delta_\tau < \delta$

$\forall \xi = \{\xi_i\}_{i=1}^{n}$, $\xi_i \in [x_{i-1}, x_i]$, $(i = 1 \div n) \Rightarrow$

$$|I - \sigma_\tau(f,\xi)| < \varepsilon$$

$I$ - интеграл на Риман на $f(x)$ в/у инт. $[a,b]$

$b$ - горна гр. на интегриране, $a$ - долна гр.

$\boxed{I = \int_a^b f(x)dx}$ – определен интеграл на Риман от $f(x)$ в/у $[a,b]$
↳ подинтегрална ф-я

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$D(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{J} \cap [0,1] \end{cases}$ не е инт. [I – безкр. много ...]

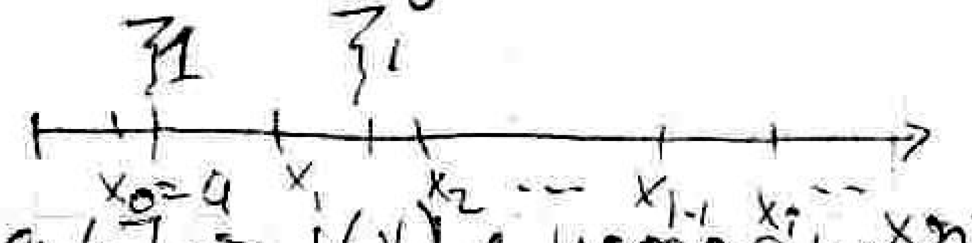
$\boxed{T - HY}$ Ако $f(x)$ – интегруема в см. на Риман в/у $[a,b]$ $\Rightarrow$ $f(x)$ е огр. в/у $[a,b]$ $\not\Leftarrow$ Дирихле

**Д-во:**

- Да докажем противното – че $f(x)$ не е огр. в/у $[a,b]$
- Т.к. $f(x)$ е инт. в/у $[a,b] \Rightarrow \exists I \in \mathbb{R} : \forall \varepsilon > 0, \exists \delta = \delta_\varepsilon > 0 :$
$\forall \tau = \{x_i\}_{i=0}^n, \delta_\tau < \delta, \forall \xi = \{\xi_i\}^n, \xi_i \in [x_{i-1}, x_i], (i = \overline{1,n}) \Rightarrow$

$$|I - \sigma_\tau(f,\xi)| < \varepsilon \Leftrightarrow I - \varepsilon < \sigma_\tau(f,\xi) < I + \varepsilon$$

- фиксираме $\varepsilon > 0$, фикс. разб. $\tau$, $\delta_\tau < \delta$



- т.к. по допускане $f(x)$ не е огр. в/у $[a,b] \Rightarrow f(x)$ е неограничена поне в/у 1 инт. $[x_{i-1}, x_i]$
ще смятаме, че не е ограничена в/у инт. $[x_0, x_1]$
$\forall \xi = \{\xi_1, \xi_2^0, \dots, \xi_n^0\}$

$$I - \varepsilon < \sigma_\tau(f, \xi') < I + \varepsilon$$

$$I - \varepsilon < f(\xi_1)\Delta x_1 + \overbrace{\sum_{i=2}^n f(\xi_i^0)\Delta x_i}^{A} < I + \varepsilon$$

$$I - A - \varepsilon < f(\xi_1)\Delta x_1 < I - A + \varepsilon, \ \forall \xi_1 \in [x_0, x_1] : \Delta x_1 > 0$$

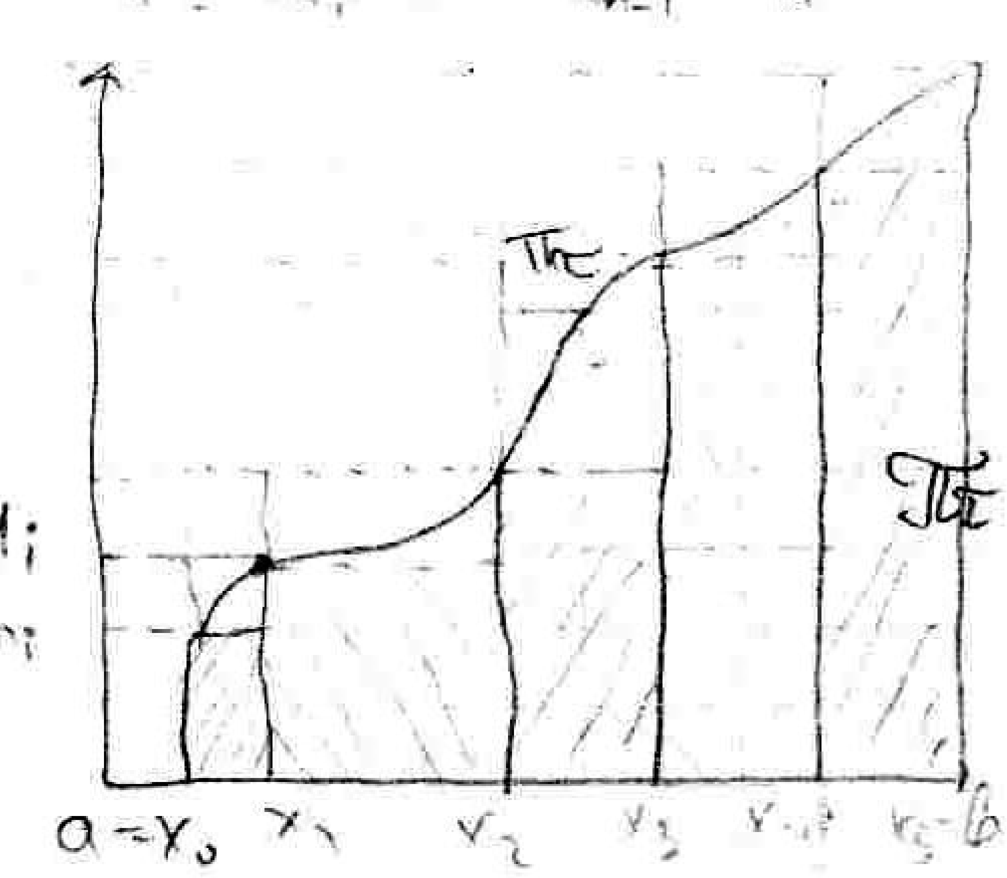
$$\frac{I - A - \varepsilon}{\Delta x_1} < f(\xi_1) < \frac{I - A + \varepsilon}{\Delta x_1}, \ \forall \xi_1 \in [x_0, x_1] \Rightarrow$$

$f(x_1)$ е огр. в/у инт. $[x_0, x_1]$ ↯ $\Rightarrow$ безкр. бр. свд. с огр. сума
$f(x)$ е огр в/у $[a,b]$

## 2. Суми на Дарбу. Критерий за интегруемост на функция.

- Нека $f(x)$ е ограничена в/у $[a,b]$
$\forall \tau = \{x_i\}_{i=1}^n$

$\Rightarrow f(x)$ е огр. в/у $\forall [x_{i-1}, x_i]$ $(i = 1 \div n)$
$\Rightarrow \forall i = 1 \div n \ \exists \sup\limits_{x \in [x_{i-1}, x_i]} f(x) = M_i$
$\exists \inf\limits_{x \in [x_{i-1}, x_i]} f(x) = m_i$



$S_\tau = \sum_{i=1}^n M_i \Delta x_i$ – голяма
$s_\tau = \sum_{i=1}^n m_i \Delta x_i$ – малка
сума на Дарбу

$\Pi_\tau \subset D \subset \Pi_\tau$
$S_\tau = S(\Pi_\tau)$
$s_\tau = S(\Pi_\tau)$
$D = \{(x,y) : a \le x \le b, 0 \le y \le f(x)\}$

Нека $f(x)$ е огр. в/у $[a,b]$

**Свойство 1**: $\forall \tau = \{x_i\}_{i=0}^{n}, \forall \xi = \{\xi_i\}_{i=1}^{n}, \xi_i \in [x_{i-1}, x_i], (\forall i = 1 \div n)$

$$\Rightarrow s_\tau \le \sigma_\tau(f;\xi) \le S_\tau$$

**Д-во:**

$\sigma_\tau(f;\xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \ \forall i = 1\div n, \ \xi_i \in [x_{i-1}, x_i], \ \xi = \{\xi_i\}_{i=1}^{n}$

$m_i \le f(\xi_i) \le M_i \quad (\forall i = 1\div n) \,|\cdot(\Delta x_i > 0)$

$m_i \Delta x_i \le f(\xi_i)\Delta x_i \le M_i \Delta x_i, \ \forall i = 1\div n$ (+)

$$\Rightarrow \underbrace{\sum_{i=1}^{n} m_i \Delta x_i}_{s_\tau} \le \underbrace{\sum_{i=1}^{n} f(\xi_i)\Delta x_i}_{\sigma_\tau(f;\xi)} \le \underbrace{\sum_{i=1}^{n} M_i \Delta x_i}_{S_\tau}$$

$$s_\tau \le \sigma_\tau(f;\xi) \le S_\tau$$

**Свойство 2**: $s_\tau = \inf \sigma_\tau(f;\xi)$

$S_\tau = \sup \sigma_\tau(f;\xi)$

**Д-во:**

? $s_\tau = \inf_{\xi} \sigma_\tau(f;\xi) \Leftrightarrow 1) \ \forall \xi: s_\tau \le \sigma_\tau(f;\xi)$ (Св. 1)

?2) $\forall \varepsilon > 0, \exists \xi_\varepsilon : \sigma_\tau(f;\xi_\varepsilon) < s_\tau + \varepsilon$

$\forall \varepsilon > 0 \Rightarrow \varepsilon_0 = \boxed{\frac{\varepsilon}{b-a}} > 0$

$m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$, то за $\varepsilon_0 > 0 \Rightarrow \exists \xi_i^{\varepsilon} \in [x_{i-1}, x_i]$

$a = x_0 \quad x_1 \quad \dots \quad x_{i-1} \quad x_i \quad \dots \quad x_n = b$

$\tau = \{x_i\}_{i=0}^{n}$

$f(\xi_i^{\varepsilon}) < m_i + \frac{\varepsilon}{b-a} \,|\cdot \Delta x_i$

$\forall i = 1\div n: f(\xi_i^{\varepsilon})\Delta x_i < m_i \Delta x_i + \frac{\varepsilon}{b-a}\Delta x_i$

$$\Rightarrow \sum_{i=1}^{n} f(\xi_i^{\varepsilon})\Delta x_i < \sum_{i=1}^{n} m_i \Delta x_i + \sum_{i=1}^{n} \frac{\varepsilon}{b-a}\Delta x_i$$

$$\sigma_i(f;\xi_\varepsilon) < s_\tau + \frac{\varepsilon}{\cancel{b-a}} \cancel{\sum_{i=1}^{n}\Delta x_i} \to b-a$$

$\sigma_\tau(f;\xi_\varepsilon) < s_\tau + \varepsilon \Rightarrow 2)$ ✓

**Свойство 3**: Ако $\tau < \tau' \Rightarrow s_\tau \le s_{\tau'} \le S_{\tau'} \le S_\tau$

**Док:**

$\tau$: $x_0 = a \quad x_1 \quad \dots \quad x_{n-1} \quad x_n = b$

$\tau'$: $x_0' \quad x_1' \quad \dots \quad x_{n-1}' \quad x_n' \quad x_{n+1}'$

$\tau = \{x_i\}_{i=0}^{n}$, $\tau' = \{x_i'\}_{i=0}^{n+1}$ $\begin{cases} x_i' = x_i \ (\forall i = 0\div n-1) \\ x_n' \in (x_{n-1}, x_n), \ x_{n+1}' = x_n \end{cases}$

$\tau: s_\tau = \sum_{i=1}^{n} m_i \Delta x_i$

$\tau': s_{\tau'} = \sum_{i=1}^{n+1} m_i' \Delta x_i'$, където $m_i' = \inf_{x\in[x_{i-1}', x_i']} f(x)$, $\Delta x_i' = x_i' - x_{i-1}'$

$\forall i = 1 \div n-1 \begin{cases} m_i' = m_i \\ \Delta x_i' = \Delta x_i \end{cases}$

$m_n^* = \inf_{x\in[x_{n-1}^*, x_n^*]} f(x)$ при инт.

$m_n' = \inf_{x\in[x_{n-1}', x_n']} f(x)$

$m_{n+1}' = \inf_{x\in[x_n', x_{n+1}']} f(x)$

$\Rightarrow m_n \le m_n'$, $m_n \le m_{n+1}'$

$\longrightarrow \Delta x_n = \Delta x_n' + \Delta x_{n+1}'$

$S_\tau = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n-1} m_i \Delta x_i + m_n \Delta x_n = \sum_{i=1}^{n-1} m_i' \Delta x_i' + m_n \Delta x_n' + m_n \Delta x_{n+1}' \le$

$\le \sum_{i=1}^{n-1} m_i' \Delta x_i' + m_n' \Delta x_n' + m_{n+1}' \Delta x_{n+1}' = \sum_{i=1}^{n+1} m_i' x_i' = S_{\tau'}$

$\Rightarrow S_\tau \le S_{\tau'}$

**Свойство 4:** $\forall \tau, \tau' \Rightarrow s_\tau \le S_{\tau'}$

Д-во:

Нека $\tau, \tau'$ - разб. на $[a,b]$ и $\tau'' = \tau \cup \tau'$ - разб. на $[a,b]$

т.к. $\tau < \tau''$, $\tau' < \tau''$ $\overset{св3}{\Rightarrow}$ $s_\tau \le s_{\tau''}$, $S_{\tau''} \le S_{\tau'}$ $\Rightarrow$ $s_\tau \le s_{\tau''} \le S_{\tau''} \le S_{\tau'}$

**Свойство 5.** $\exists \sup s_\tau = \underline{I}$ ← долен
и $\exists \inf S_\tau = \overline{I}$ ← горен
интеграл на Дарбу

При това $s_\tau \le \underline{I} \le \overline{I} \le S_\tau$ $(\forall \tau)$

Д-во:

От св. 4 $\Rightarrow \forall \tau, \tau' \Rightarrow s_\tau \le S_{\tau'}$

фиксираме $\tau' \Rightarrow \{s_\tau : \tau\}$ е огр. отгоре

$\overset{\text{П-п за непр.}}{\Rightarrow}$ $\exists \sup_\tau s_\tau = \underline{I}$ (1)

$\tau'$ - произв. фикс. $\Rightarrow \underline{I} \le S_{\tau'}$, $\forall \tau'$

$\Rightarrow \{S_{\tau'} ; \tau'\}$ - огр. отдолу $\Rightarrow \exists \inf_{\tau'} S_{\tau'} = \overline{I}$ (2)

$\overset{1,2}{\Rightarrow} \underline{I} \le \overline{I}$

**Т: Критерий за интегруемост:** $f(x)$ - деф и огр. в/у $[a,b]$

Ф-та $f(x)$ е интегруема по Риман в/у $[a,b] \Leftrightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$:

$\forall \tau = \{x_i\}_{i=0}^n : \delta_\tau < \delta \Rightarrow S_\tau - s_\tau < \varepsilon$

Д-во:

$\Rightarrow$) $f(x)$ - инт. в/у $[a,b] \Rightarrow \exists I \in \mathbb{R} : \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) : \forall \tau = \{x_i\}_{i=0}^n$,

$\delta_\tau < \delta$, $\forall \xi = \{\xi_i\}_{i=1}^n$, $\xi_i \in [x_{i-1}, x_i]$, $(\forall i = 1 \div n) \Rightarrow$

$|I - \sigma_\tau(f,\xi)| < \frac{\varepsilon}{3}$ *

$* \Leftrightarrow I - \frac{\varepsilon}{3} \overset{1}{<} \sigma_\tau(f;\xi) \overset{2}{<} I + \frac{\varepsilon}{3}$ ; $(\forall \xi)$

1) $I - \frac{\varepsilon}{3} < \sigma_\tau(f,\xi)$ $\forall \xi \Rightarrow \inf_\xi \sigma_\tau(f,\xi) \ge I - \frac{\varepsilon}{3}$

$I - \frac{\varepsilon}{3} \le s_\tau$ при $\tau$ фикс.

2) $\sigma_\tau(f,\xi) < I + \frac{\varepsilon}{3}$ $(\forall \xi)$

$\Rightarrow \sup \sigma_\tau(f;\xi) \le I + \frac{\varepsilon}{3}$

$S_\tau \le I + \frac{\varepsilon}{3}$

$\overset{1,2}{\Rightarrow} \begin{cases} S_\tau \le I + \frac{\varepsilon}{3} \\ -s_\tau \le -I + \frac{\varepsilon}{3} \end{cases} (+) \Rightarrow S_\tau - s_\tau \le \frac{2\varepsilon}{3} < \varepsilon$

$\Leftarrow$ | Нека $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=0}^n, \delta_\tau < \delta \Rightarrow$

$S_\tau - s_\tau < \varepsilon$ ★

$\underline{I} = \sup\limits_\tau s_\tau$

$\overline{I} = \inf\limits_\tau S_\tau$ $\quad s_\tau \le \underline{I} \le \overline{I} \le S_\tau \\ -\underline{I} \le -s_\tau \Big] + \Rightarrow 0 \le \overline{I} - \underline{I} \le S_\tau - s_\tau, \forall \tau$

★ $\Rightarrow \forall \varepsilon > 0, \exists \tau : S_{\tau_\varepsilon} - s_{\tau_\varepsilon} < \varepsilon$

$0 \le \overline{I} - \underline{I} \le S_{\tau_\varepsilon} - s_{\tau_\varepsilon} < \varepsilon$

$\Rightarrow \forall \varepsilon > 0 : 0 \le \overline{I} - \underline{I} < \varepsilon \Rightarrow \overline{I} - \underline{I} = 0 \Leftrightarrow \overline{I} = \underline{I} = I$

т.е. $\forall \tau : \underset{s_\tau \le}{I} \le S_\tau$

$\forall \tau, \forall \xi = \{\xi_i\}_{i=1}^n \quad s_\tau \le \sigma_\tau(f,\xi) \le S_\tau \Big] \Rightarrow \begin{matrix} \forall \tau, \forall \xi = \{\xi_i\}_{i=1}^n \\ s_\tau \le I \le S_\tau \\ -S_\tau \le -\sigma(f,\xi) \le -s_\tau \end{matrix} \Big] (+)$

$\Rightarrow -(S_\tau - s_\tau) \le I - \sigma_\tau(f,\xi) \le S_\tau - s_\tau$

$\Rightarrow |I - \sigma_\tau(f,\xi)| \le S_\tau - s_\tau, \forall \tau, \forall \xi$

(усл.) $\forall \varepsilon > 0 \; \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=1}^n, \delta_\tau < \delta, \forall \xi = \{\xi_i\}_{i=1}^n$

$\Rightarrow |I - \sigma_\tau(f,\xi)| \le S_\tau - s_\tau < \varepsilon$

$\Rightarrow f(x)$ е инт. по Риман в/у $[a,b]$

∘ Следствие: $\int_a^b f(x)\,dx = I = \overline{I} = \underline{I} = \sup\limits_\tau s_\tau = \inf\limits_\tau S_\tau$

∘ Функция на Дирихле: $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{I} \cap [0,1] \end{cases}$

Д-во за интегруемост:

$\forall \tau = \{x_i\}_{i=0}$ на $[0,1]$ $\quad$ $0 \; - - \; x_{i-1} \; x_i \; - - \; 1$

$- s_\tau = \sum\limits_{i=1}^n m_i \Delta x_i, \; \forall i = 1 \div n, \; m_i = \inf\limits_{x \in [x_{i-1}, x_i]} D(x) \overset{\text{т-п за ирр.}}{=} 0$

$\Rightarrow s_\tau = 0, \; \underline{I} = \sup s_\tau = 0$

$- S_\tau = \sum\limits_{i=1}^n M_i \Delta x_i, \; \forall i = 1 \div n, \; M_i = \sup\limits_{x \in [x_{i-1}, x_i]} D x = 1$

$\Rightarrow \overline{I} = \inf S_\tau = 1$

$\underline{I} = 0 < 1 = \overline{I}, \; \underline{I} \ne \overline{I} \Rightarrow D(x)$ не е инт. в/у $[0,1]$

Деф | Нека $f(x)$ - деф. и огр. в/у $[a,b]$. Колебание на $f(x)$ в/у $[a,b]$ наричаме числото - $\boxed{\omega(f) = M - m}$, където $M = \sup\limits_{x \in [a,b]} f(x)$, $m = \inf\limits_{x \in [a,b]} f(x)$

Тв | Нека $f(x)$ е огр. в/у $[a,b]$. Тогава $\omega(f) = \sup |f(x') - f(x'')|$, $x', x'' \in [a,b]$

Д-во:

$\omega(f) = \sup |f(x') - f(x'')| \Leftrightarrow$ 1) $\forall x', x'' \in [a,b] \rightarrow |f(x') - f(x'')| \le \omega(f)$;
2) $\forall \varepsilon > 0, \exists \bar{x}', \bar{x}'' \in [a,b] \rightarrow |f(\bar{x}') - f(\bar{x}'')| > \omega(f) - \varepsilon$

1) Нека $x_{min} \in [a,b]: f(x_{min}) = \inf\limits_{x\in[a,b]} f(x)$ (*)

Нека $x_{max} \in [a,b]: f(x_{max}) = \sup\limits_{x\in[a,b]} f(x)$ (**)

$$\Rightarrow \begin{cases} f(x_{min}) \le f(x') \le f(x_{max}) & \forall x' \in [a,b] \\ -f(x_{min}) \ge -f(x'') \ge -f(x_{max}) & \forall x'' \in [a,b] \end{cases} \Big| +$$

$$\Rightarrow |f(x') - f(x'')| \le f(x_{max}) - f(x_{min}) = \sup\limits_{x\in[a,b]} f(x) - \inf\limits_{x\in[a,b]} f(x) = \omega(f)$$

2) $\overset{*}{\Rightarrow} \forall \frac{\varepsilon}{2} > 0,\ \exists \bar{x}' \in [a,b]: f(x_{min}) \le f(\bar{x}') < f(x_{min}) + \frac{\varepsilon}{2}$ (1)

$\overset{**}{\Rightarrow} \forall \frac{\varepsilon}{2} > 0,\ \exists \bar{x}'' \in [a,b]: f(x_{max}) \ge f(\bar{x}'') > f(x_{max}) - \frac{\varepsilon}{2}$ (2)

$\overset{1-2}{\Rightarrow} f(\bar{x}') - f(\bar{x}'') < f(x_{min}) - f(x_{max}) + \varepsilon$ (3)

$\overset{2-1}{\Rightarrow} f(\bar{x}'') - f(\bar{x}') > f(x_{max}) - f(x_{min}) - \varepsilon$ (4)

$\overset{3\&4}{\Rightarrow} |f(\bar{x}') - f(\bar{x}'')| > f(x_{max}) - f(x_{min}) - \varepsilon = \omega(f) - \varepsilon$

**Т - критерий за интегруемост:**

Нека $f(x)$ - огр. в/у $[a,b]$

ф-ця $f(x)$ - инт. в/у $[a,b] \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0.$

$\forall \tau = \{x_i\}_{i=0}^{n},\ \delta\tau = \delta \Rightarrow \boxed{\sum\limits_{i=1}^{n} \omega_i \Delta x_i < \varepsilon}$

## 3. Класове интегруеми функции

**Т** | Нека $f(x)$ - <u>непр.</u> в/у $[a,b]$. Тогава $f(x)$ - инт. в/у $[a,b]$

Д-во:

Т.к. $f(x)$ е непр. в/у $[a,b] \Rightarrow$ равном. непр в/у $[a,b] \Rightarrow$

$\forall \varepsilon > 0,\ \varepsilon' = \frac{\varepsilon}{2(b-a)} > 0$, т.к. $f(x)$ - равном. н. в/у $[a,b]$, то $\exists \delta = \delta(\varepsilon) > 0, \forall x', x''$

$\in [a,b],\ |x' - x''| < \delta \Rightarrow |f(x') - f(x'')| < \varepsilon' = \frac{\varepsilon}{2(b-a)}$

$\forall \tau = \{x_i\}_{i=1}^{n} : \delta\tau < \delta$

$$S_\tau - s_\tau = \sum_{i=1}^{n} \omega_i(f) \Delta x_i \le \sum_{i=1}^{n} \frac{\varepsilon}{2(b-a)} \Delta x_i = \frac{\varepsilon}{2(b-a)} \cdot \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{2\cancel{(b-a)}} \cancel{(b-a)} < \varepsilon$$

$$\omega_i(f) = \sup_{x', x'' \in [x_{i-1}, x_i]} |f(x') - f(x'')| \le \sup_{x', x'' \in [x_{i-1}, x_i]} \varepsilon' = \varepsilon' = \frac{\varepsilon}{2(b-a)}$$

$|x' - x''| \le |[x_{i-1}, x_i]| \le \delta\tau < \delta$

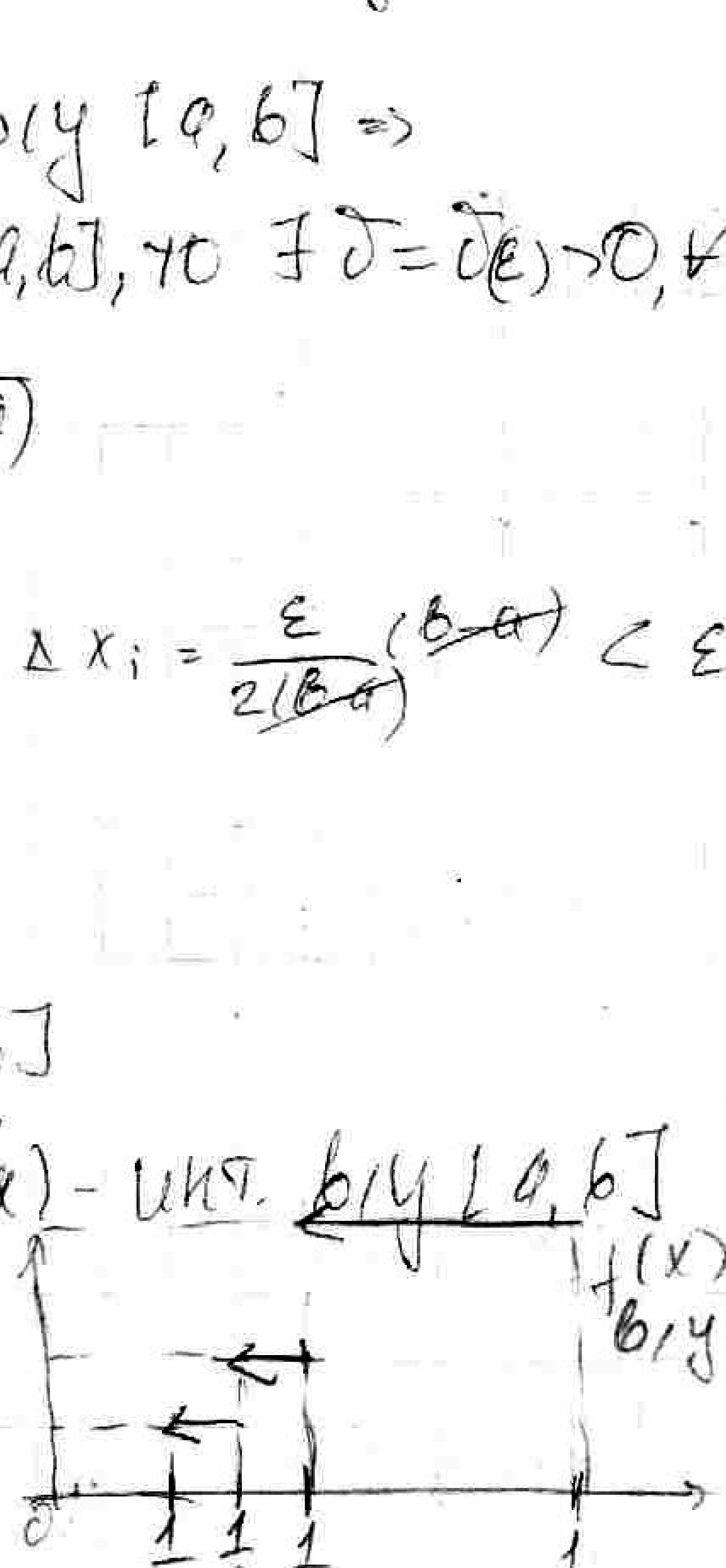
$S_\tau - s_\tau < \varepsilon \Rightarrow$ (кр. за инт.) $f(x)$ е инт. в/у $[a,b]$

**Т** | Нека $f(x)$ - <u>монотонна</u> в/у $[a,b] \Rightarrow f(x)$ - инт. <u>в/у $[a,b]$</u>

• Постр. $f(x) = \begin{cases} \frac{1}{n}, & x \in (\frac{1}{n+1}, \frac{1}{n}],\ n \in \mathbb{N} \\ 0, & x = 0 \end{cases}$

$D(f) = [0,1]$

?• $\int_0^1 f(x) =$




$f(x)$ - м.р. в/у $[0,1]$

Д-во:

Нека $f(x)$ - м.р. в/у $[a,b]$
$\tau$ - разб., $\tau = \{x_i\}_{i=0}^{n}$, р. на $[a,b]$
$\forall x \in [x_{i-1}, x_i]$

$$\Rightarrow f(x_{i-1}) \le f(x) \le f(x_i)$$
$$m_i = \inf_{x\in[x_{i-1},x_i]} f(x) = f(x_{i-1})$$
$$M_i = \sup_{x\in[x_{i-1},x_i]} f(x) = f(x_i)$$

$$S_\tau - s_\tau = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \Delta x_i$$

$\forall \varepsilon > 0, \delta = \delta(\varepsilon) = \frac{\varepsilon}{2[f(b)-f(a)]} > 0.$

$\forall \tau = \{x_i\}_{i=0}^{n} : \delta_\tau < \delta = \frac{\varepsilon}{2(f(b)-f(a))}$

$$\Rightarrow S_\tau - s_\tau = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x_i < \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\delta = \delta \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

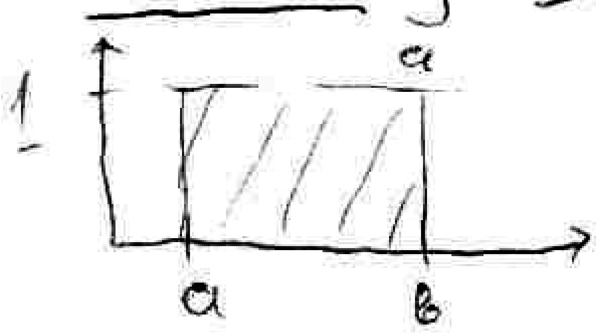
$$= \frac{\varepsilon}{2[f(b)-f(a)]} \cdot f(b) - f(a) = \frac{\varepsilon}{2} < \varepsilon.$$

$\Rightarrow$ (кр. за инт.) $f(x)$ е инт. в/у $[a,b]$

**Тh** $f(x)$ - огр. в/у $[a,b]$ и има кр. бр. т. на прекъсване, тогава $f(x)$ - инт. в/у $[a,b]$

## 4 Свойства на определения интеграл

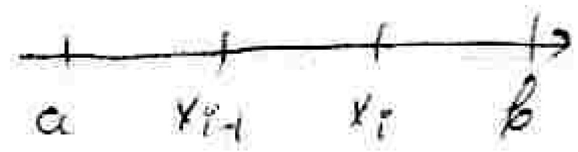
**Св. 1** $\int_a^b 1\, dx = b-a$



Д-во:

$\int_a^b 1\, dx = \sup_\tau s_\tau$ *

Нека $\tau = \{x_i\}_{i=0}^{n}$ - разбиване на $[a,b]$



$\forall i = 1 \div n : m_i = \inf_{x\in[x_{i-1},x_i]} f(x) = \inf_{x\in[x_{i-1},x_i]} 1 = 1$

$\Rightarrow s_\tau = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 1\, \Delta x_i = b-a$

$* \Rightarrow \sup_\tau (b-a) = b-a = \int_a^b 1\, dx$

**Деф** $\int_a^a f(x)\, dx = 0$

**Деф** Ако $a < b$, то $\int_b^a f(x)\, dx := -\int_a^b f(x)\, dx$

**Св 2** Нека $f(x)$ и $g(x)$ - интегр. в/у $[a,b]$, $\lambda \in \mathbb{R}$

$\Rightarrow f(x) + g(x)$, $\lambda f(x)$ са интегр. в/у $[a,b]$

1) $\int_a^b [f(x)+g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

2) $\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx$

Следствие 1: При усл. на св. 2 $f(x)-g(x)$ е инт. в/у $[a,b]$ и
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Д-во:

1) Нека $\tau=\{x_i\}_{i=0}^n$ - разб. на $[a,b]$, $\xi=\{\xi_i\}_{i=1}^n$, $\xi_i \in [x_{i-1}, x_i]$ $(\forall i = 1\div n)$

$$\sigma_\tau(f+g;\xi) = \sum_{i=1}^n (f+g)(\xi_i)\Delta x_i = \sum_{i=1}^n [f(\xi_i)+g(\xi_i)]\Delta x_i =$$
$$= \sum_{i=1}^n f(\xi_i)\Delta x_i + \sum_{i=1}^n g(\xi_i)\Delta x_i = \sigma_\tau(f,\xi) + \sigma_\tau(g;\xi)$$

Т.к. $f(x)$ и $g(x)$ - инт. в/у $[a,b] \Rightarrow \exists I_1, I_2 : \forall \varepsilon>0, \exists \delta=\delta(\varepsilon)>0:$ $\Big]\star$
$\forall \tau=\{x_i\}_{i=1}^n$, $\delta_\tau<\delta$, $\forall \xi=\{\xi_i\}_{i=1}^n$

$|I_1 - \sigma_\tau(f,\xi)| < \frac{\varepsilon}{2}$ (1)

$|I_2 - \sigma_\tau(g,\xi)| < \frac{\varepsilon}{2}$

Разгл. $|\sigma_\tau(f+g,\xi) - (I_1+I_2)| = |\sigma_\tau(f,\xi) + \sigma_\tau(g,\xi) - (I_1+I_2)| \le$
$$|\sigma_\tau(f,\xi) - I_1| + |\sigma_\tau(g;\xi) - I_2| \overset{(1)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow$ за $f+g$, $\exists I_1+I_2 : \star \Rightarrow (2)$, т.е. $(f+g)(x)$ е инт. в/у $[a,b]$ и
$$\int_a^b [f(x)+g(x)]\,dx = I_1+I_2 = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2) $\exists I_1 : \forall \varepsilon>0, \exists \delta=\delta(\varepsilon)>0 : \forall \tau=\{x_i\}_0^n, \forall \xi \Rightarrow$
$$|\sigma_\tau(f,\xi) - I_1| < \frac{\varepsilon}{|\lambda|} \;(3) \quad (\lambda \ne 0) \quad (\text{за } \lambda=0 \text{ е изп.})$$
$$\sigma_\tau(\lambda f,\xi) = \lambda\,\sigma_\tau(f,\xi)$$
$$\Rightarrow |\sigma(\lambda f,\xi) - \lambda I_1| = |\lambda\,\sigma_\tau(f,\xi) - \lambda I_1| = |\lambda|\,|\sigma_\tau(f,\xi) - I_1| \overset{(3)}{<} |\lambda| \frac{\varepsilon}{|\lambda|} = \varepsilon$$

$\Rightarrow \lambda f$ е инт. в/у $[a,b]$ и $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$

$\overset{1,2}{\Longrightarrow}$ $R[a,b] = \{f(x) :$ инт. по Риман в/у $[a,b]\}$ е л.п.
$\int_a^b f(x)\,dx : R[a,b] \to R$ – л. оп., линеен функционал

Св. 3 Ако $f(x)$ е инт. в/у $[a,b]$ и неотр. в/у $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$

Д-во:

$\int_a^b f(x)\,dx = \sup_\tau s_\tau$ $(\star)$

$s_\tau = \sum_{i=1}^n m_i \cdot \Delta x_i$ и $m_i = \inf_{x\in[x_{i-1},x_i]} f(x) \ge 0$, $(\forall i = 1\div n)$

$\Rightarrow s_\tau \ge 0 \Rightarrow \sup_\tau s_\tau \ge 0 \overset{\star}{\Rightarrow} \int_a^b f(x)\,dx \ge 0$

Следствие 2 Ако $f(x)$ и $g(x)$ - инт. и $f(x) \ge g(x)$ в/у $[a,b] \Rightarrow$
$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Д-во:

т.к. $f(x)-g(x)$ в/у $[a,b] \Rightarrow f(x)-g(x) \ge 0$ в/у $[a,b] \overset{св. 3}{\Rightarrow}$
$$\int_a^b [f(x)-g(x)]\,dx \ge 0 \quad \underset{сл. 1}{=} \quad \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

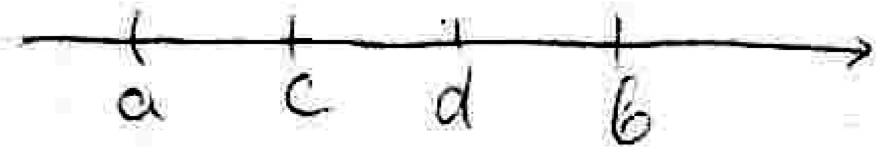
Св. 4] Ако $f(x)$ - инт. в/у $[a,b] \Rightarrow |f(x)|$ инт в/у $[a,b]$ и
$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Д-во:

$|f(x)| \ge f(x)$ и $|f(x)| \ge -f(x) \xRightarrow{\text{сл.1}}$
$$\int_a^b |f(x)|\,dx \ge \int_a^b f(x)\,dx \text{ и } \int_a^b |f(x)|\,dx \ge -\int_a^b f(x)\,dx$$
$$\Rightarrow \int_a^b |f(x)|\,dx \ge \left|\int_a^b f(x)\,dx\right|$$

Св. 5] Ако $f(x)$ е инт. в/у $[a,b] \Rightarrow f(x)$ е инт. в/у всеки подинт. $[c,d] \subset [a,b]$

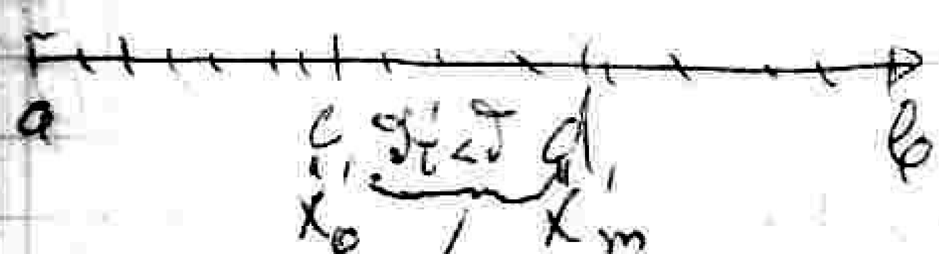
Д-во:



$f(x)$ е инт. в/у $[a,b] \Rightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0,$
$\forall \tau = \{x_i\}_{i=0}^n, \delta_\tau < \delta \Rightarrow \sum_{i=1}^n w_i(f).\Delta x_i < \varepsilon$

Нека има произв. разбиване:

$\forall \tau' = \{x_i\}_{i=0}^m$ на инт. $[c,d]$ : $\delta_{\tau'} < \delta$

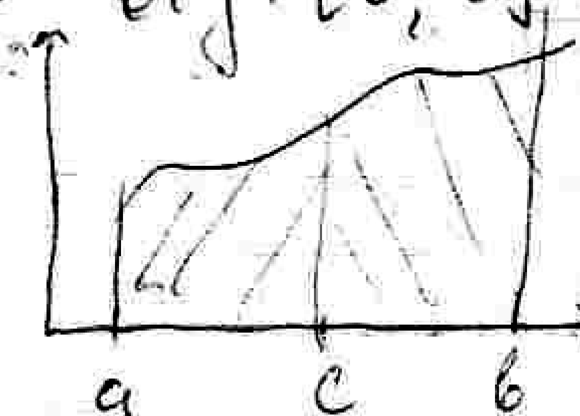


$\Rightarrow \exists \tau = \{x_i\}_{i=0}^n > \tau'$: $\delta_\tau < \delta$

$$\sum_{i=0}^m w_i(f)\Delta x_i' \le \sum_{i=1}^n w_i(f)\Delta x_i < \varepsilon$$
$\Delta x_i' = x_i'' - x_{i-1}'$ $(\forall i = 1 \div n)$

$\xRightarrow{\text{кр.заинт.}}$ $f(x)$ е също инт. в/у $[c,d]$

Свойство 6:] Ако $f(x)$ е инт. в/у $[a,b]$ и $c \in [a,b] \Rightarrow$
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



-> за лицата

Д-во:

Нека $c \in [a,b]$ $a < c < b$

$\xRightarrow{\text{св.5}}$ $f(x)$ е инт. в/у $[a,c]$ и $[c,b]$

Нека $I = \int_a^b f(x)\,dx$, $I_1 = \int_a^c f(x)\,dx$, $I_2 = \int_c^b f(x)\,dx$

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ :

1) $\forall \tau^{[a,b]} = \{x_i\}_{i=0}^n$ : $\delta_{\tau^{[a,b]}} < \delta \Rightarrow$
$\forall \xi^{[a,b]} = \{\xi\}_{i=1}^n \Rightarrow |I - \sigma_{\tau^{[a,b]}}(f, \xi^{[a,b]})| < \varepsilon$

2) $\forall \tau^{[a,c]} = \{x_i'\}_{i=0}^m$ : $\delta_{\tau^{[a,c]}} < \delta$
$\forall \xi^{[a,c]} \Rightarrow |I_1 - \sigma_{\tau^{[a,b]}}(f, \xi^{[a,b]})| < \varepsilon$

3) $\forall \tau^{[c,b]} = \{x_i''\}_{i=0}^k$ : $\delta_{\tau^{[a,b]}} < \delta$ $\forall \xi^{[a,b]} \Rightarrow$
$\forall \xi^{[c,b]} \Rightarrow |I_2 - \sigma_{\tau^{[c,b]}}(f, \xi^{[c,b]})| < \varepsilon$

Нека $\tau^{[a,c]}: \delta_\tau[a,c] < \delta$ и $\tau^{[c,b]}: \delta_\tau[c,b] < \delta$

$\xi_0^{[a,c]}, \xi_0^{[c,b]}, \xi_0^{[a,b]} = \xi_0^{[a,c]} \cup \xi_0^{[c,b]}$ – фиксирани са ; $\tau^{[a,b]} = \tau^{[a,c]} \cup \tau^{[c,b]}$, $\delta_\tau[a,b] < \delta \ (*)$,

$\Rightarrow$ за $(*)$ ще са в сила с-вата за $I$ от 1, 2, 3

$$\Rightarrow |I-(I_1+I_2)| = \left|I - \sigma_{\tau^{[a,b]}}(f;\xi_0^{[a,b]}) + \left[\sigma_{\tau^{[a,b]}}(f;\xi_0^{[a,b]}) - (I_1+I_2)\right]\right|$$

$$\le \left|I - \sigma_{\tau^{[a,b]}}(f,\xi_0^{[a,b]})\right| + \left|\sigma_{\tau^{[a,c]}}(f,\xi_0^{[a,c]}) + \sigma_{\tau^{[c,b]}}(f,\xi_0^{[c,b]}) - (I_1+I_2)\right|$$

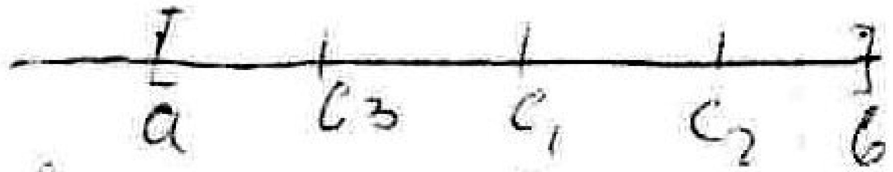
$$\le \left|I - \sigma_{\tau^{[a,b]}}(f,\xi_0^{[a,b]})\right| + \left|I_2 - \sigma_{\tau^{[c,b]}}(f,\xi_0^{[c,b]})\right| + \left|I_1 - \sigma_{\tau^{[a,c]}}(f;\xi_0^{[a,c]})\right|$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

т.е. $\forall \varepsilon > 0 \Rightarrow |I-(I_1+I_2)| < 3\varepsilon \Rightarrow I-(I_1+I_2) = 0 \Leftrightarrow I = I_1 + I_2 \Leftrightarrow$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>Следствие 4:</u>




$$\int_{c_1}^{c_3} f(x)\,dx \overset{?}{=} \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx$$

$$\int_{c_3}^{c_2} f(x)\,dx = \int_{c_3}^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx \ (\text{подредба}) \Rightarrow \int_{c_3}^{c_1} f(x)\,dx = \int_{c_1}^{c_2} f(x)\,dx - \int_{c_3}^{c_2} f(x)\,dx \Rightarrow$$

$$\int_{c_1}^{c_3} f(x)\,dx = \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx \,!$$

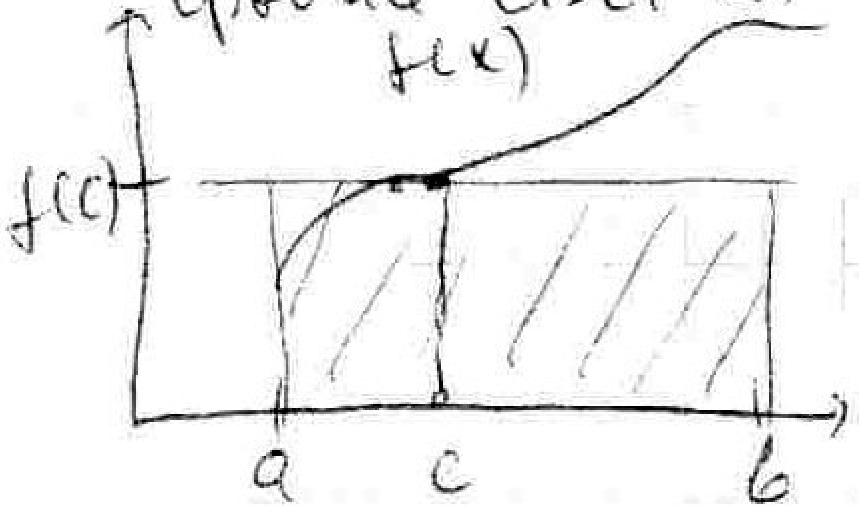
Аналог. за Др. 5

<u>Свойство 7:</u> Нека $f(x)$ непр. в/у $[a,b] \Rightarrow \exists\, c \in [a,b]$:

$$\boxed{\int_a^b f(x)\,dx = f(c)\cdot(b-a)}$$

- $f(c) = \frac{1}{b-a}\cdot\int_a^b f(x)\,dx$ (теорема за средната стойност)

средна ст-ст на $f(x)$



<u>Д-во</u>:

Т.к. $f(x) \in$ непр. в/у $[a,b] \overset{\text{т.В.}}{\Rightarrow} \exists\, x_0, x_1 \in [a,b]$:

$\forall\, x \in [a,b]: f(x_0) \le f(x) \le f(x_1) \Rightarrow$

$\int_a^b f(x_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(x_1)\,dx \Rightarrow$

$f(x_0)(b-a) \le \int_a^b f(x)\,dx \le f(x_1)(b-a) \,:\, b-a \ne 0 \Rightarrow$

$f(x_0) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(x_1) \overset{\text{т.м.с.}}{\Rightarrow} \exists\, c \in [a,b] \Rightarrow f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$, т.е.:

$\int_a^b f(x)\,dx = f(c)(b-a)$

# 3. Интеграл с променлива горна граница. Формула на Нютон-Лайбниц.

Деф. Нека $f(x)$ е инт. в/у $[a,b] \Rightarrow \forall x \in [a,b] \Rightarrow f$ е инт. в/у $[a,x] \Rightarrow$

$\exists \int_a^x f(t)\,dt,\ x \in [a,b] \Rightarrow F(x) = \int_a^x f(t)\,dt,\ x \in [a,b]$ интегр. като ф-я на горната и гр.

$G(x) = \int_x^b f(t)\,dt,\ x \in [a,b]$ — " — на долната гр. граница

<u>Свойство 1</u> (теорема) Ако $f(x)$ е инт. в/у $[a,b] \Rightarrow F(x) = \int_a^x f(t)\,dt$ е непр. в/у $[a,b]$.

Д-во:

Нека $x \in [a,b]$

$? \lim_{\Delta x \to 0} F(x+\Delta x) = F(x)\ ? \Leftrightarrow \lim_{\Delta x \to 0} [F(x+\Delta x) - F(x)] = 0\ ?$

$F(x+\Delta x) - F(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt = \cancel{\int_a^x f(t)\,dt} + \int_x^{x+\Delta x} f(t)\,dt - \cancel{\int_a^x f(t)\,dt}$ (св. 4) $= \int_x^{x+\Delta x} f(t)$

Нека $|F(x+\Delta x) - F(x)| = \left|\int_x^{x+\Delta x} f(t)\,dt\right| \le \left|\int_x^{x+\Delta x} |f(t)|\,dt\right|$

$f(x)$ - инт. в/у $[a,b] \Rightarrow$ е огр. в/у $[a,b] \Rightarrow$

$\exists M > 0 : |f(x)| \le M\ \forall x \in [a,b] \Rightarrow$

$\left|\int_x^{x+\Delta x} |f(t)|\,dt\right| \le M \left|\int_x^{x+\Delta x} 1\,dx\right| = M|\Delta x|$

$0 \le |F(x+\Delta x) - F(x)| \le M|\Delta x|$

$\Delta x \to 0$: $0$, $0$

$\Rightarrow \lim_{\Delta x \to 0} [F(x+\Delta x) - F(x)] = 0 \Rightarrow F(x)$ - непр. във $\forall$ т. $x \in [a,b]$

<u>Свойство 2:</u> (теорема) Ако $f(x)$ е инт. в/у $[a,b]$ и $f(x)$ - непр. в т. $x_0 \in [a,b]$ $\Rightarrow F(x)$ е диф в т. $x_0$. При това $F'(x_0) = f(x_0)$

Д-во:

$\exists\ ?\ F'(x_0) = f(x_0) \Leftrightarrow \lim_{x \to x_0} \frac{\Delta F}{\Delta x} = f(x_0)\ ?$ ? dt

$\frac{\Delta F}{\Delta x} - f(x_0) = \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} f(x)\,dt - f(x_0)\ (\Delta x : x_0 + \Delta x \in [a,b]) =$

$= \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} f(x)\,dt - \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} f(x_0)\,dt = \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} [f(t) - f(x_0)]\,dt \Rightarrow$

$\forall \varepsilon > 0$, т.к. $f(x)$ е непр. в т. $x_0$, то $\exists \delta = \delta(\varepsilon) > 0$:

$\forall \Delta x : x_0 + \Delta x_0 \in [a,b],\ |\Delta x| < \delta \Rightarrow |f(x_0+\Delta x)| < \varepsilon \Leftrightarrow$

$x \in [a,b] : |\underbrace{x - x_0}_{\Delta x}| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\left|\frac{\Delta F}{\Delta x} - f(x_0)\right| = \left|\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} [f(t) - f(x_0)]\,dt\right| = \frac{1}{|\Delta x|} \cdot \left|\int_{x_0}^{x_0+\Delta x} [f(t) - f(x_0)]\,dt\right| \le$

$\frac{1}{\Delta x}\left|\int_{x_0}^{x_0+\Delta x} |f(t) - f(x_0)|\,dt\right|$ ★

$|\Delta x| < \delta$

- $t \in [x_0, x_0+\Delta x]$ $\Delta x > 0$ $\quad |\Delta x| < \delta \Rightarrow |t - x_0| < \delta$
  $t \in [x_0+\Delta x, x_0]$ $\Delta x < 0$ $\quad |f(t) - f(x_0)| < \varepsilon \Rightarrow$

$$* \le \frac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} \varepsilon\, dt\right| = \frac{1}{|\Delta x|}\cdot \varepsilon \left|\int_{x_0}^{x_0+\Delta x} 1\, dt\right| = \frac{1}{\cancel{|\Delta x|}}\cdot \varepsilon\, \cancel{|\Delta x|} = \varepsilon$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x_0) \Rightarrow \exists\ F'(x_0) = f(x_0)$$

<u>следствие:</u> Ако $f(x)$ е непр. в/у $[a,b] \Rightarrow F(x) = \int_a^x f(x)\,dx$ е диф в/у $[a,b]$. При това $F'(x) = f(x)$, $\forall x \in [a,b]$ т.е. $F(x) = \int_a^x f(x)\,dx$ е прим. ф-я на $f(x)$ в/у $dx$.

<u>Т (Нютон-Лайбниц)</u> Нека $f(x)$ - непр. в/у $[a,b]$ и $\Phi(x)$ - прим. ф-я на $f(x)$ в/у $[a,b] \Rightarrow$

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a) = \Phi(x)\Big|_a^b$$

Пример: $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

Д-во:

$F(x) = \int_a^x f(t)\,dt$ е също е прим. на $f(x)$ в/у $[a,b] \Rightarrow$

$\exists\ c \in \mathbb{R}: \Phi(x) = F(x) + c\ (\forall x \in [a,b]) \Rightarrow$

$\Phi(a) = F(a) + c = 0 + c = c$

$F(a) = \int_a^a f(t)\,dt = 0$

$\Phi(b) = F(b) + \Phi(a) = \int_a^b f(t)\,dt + \Phi a \Rightarrow \int_a^b f(t)\,dt = \Phi(b) - \Phi(a)$

## 6. Интегриране по части и смяна на пром. в опр. инт.

<u>Т</u> Нека $f(x)$ - непр. в/у $[A,B]$ и $\varphi(t)$ е деф в/у $[\alpha,\beta]$ със ст-сти в $[A,B]$, т.е. $\varphi\,[\alpha,\beta] \to [A,B]$, такава, че:

1) $\varphi(\alpha) = a$, $\varphi(\beta) = b$

2) $\exists\ \varphi'(t)$ - непр. в/у $[\alpha,\beta]$

$$\Rightarrow \int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\cdot \underbrace{\varphi'(t)\cdot dt}_{d\varphi(t)} = \int_\alpha^\beta f(\varphi(t))\,d\varphi(t)$$

Д-во:

Нека $F(x)$ - прим. ф-я на $f(x)$ в/у $[A,B] \Rightarrow$

$\Rightarrow F(\varphi(t))$ е прим. ф-я на $f(\varphi(t))\cdot\varphi'(t)$ в/у $[\alpha,\beta]$

$\forall t \in [\alpha,\beta]: [F(\varphi(t))]' = F'(\varphi(t))\cdot\varphi'(t) = f(\varphi(t))\cdot\varphi'(t)$

$$\overset{\text{Ф.Н-Л.}}{=} \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt = F(\varphi(t))\Big|_\alpha^\beta = F(\varphi(\beta)) - F(\varphi(\alpha)) = F(b) - F(a) \overset{\text{Ф.Н-Л.}}{=}$$

$$= \int_a^b f(x)\,dx$$

<u>Т</u> Нека $f(x)$ и $g(x)$ имат непр. произв. $f'(x)$ и $g'(x)$ в/у $[a,b] \Rightarrow$

$$\int_a^b f(x)\,dg(x) = f(x)\cdot g(x)\Big|_a^b - \int_a^b g(x)\,df(x)$$

$\mathcal{Z}$-во:

$f(x).g(x)$ – има непр. произв. в/у $[a,b]$, т.к. $(f(x).g(x))' = f'(x)g(x) + f(x)g'(x)$ $\forall x \in [a,b]$

$\overset{Н.Л.}{\Rightarrow} \int_a^b (f(x)g(x))' dx = \int_a^b [f'(x)g(x) + g'(x)f(x)] dx = \int_a^b f'(x) g(x) dx + \int_a^b g'(x) f(x) =$

$= \int_a^b g(x) d f(x) + \int_a^b f(x) d g(x)$

$\Rightarrow \int_a^b (f(x).g(x))' dx = f(x).g(x) \Big|_a^b - \int_a^b g(x) d f(x)$

Примери 1) $\int_0^1 \sqrt{1-x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\, d \sin t = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t} \cos t\, dt = \int_0^{\frac{\pi}{2}} |\cos t| \cos t\, dt =$

$= \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} dt = \frac{1}{2} [ \int_0^{\pi/2} 1 dt + \int_0^{\frac{\pi}{2}} \cos 2t\, dt ] =$

$= \frac{1}{2} [\frac{\pi}{2} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2t\, d2t] = \frac{1}{2} [\frac{\pi}{2} + \frac{1}{2} \sin 2t \Big|_0^{\frac{\pi}{2}}] = \frac{1}{2} [\frac{\pi}{2} + \underbrace{\frac{1}{2} \sin \frac{2\pi}{2} - \frac{1}{2} \sin \frac{2 \cdot 0}{2}}_{0}]$

$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$, където $x = \sin t$

$x = \varphi(t) = \sin t$, $t \in [0, \frac{\pi}{2}]$, $\alpha = 0 \Rightarrow \varphi(\alpha) = \varphi(0) = \sin 0 = 0$

$\beta = \frac{\pi}{2} \Rightarrow \varphi(\beta) = \varphi(\frac{\pi}{2}) = \sin \frac{\pi}{2} = 1$

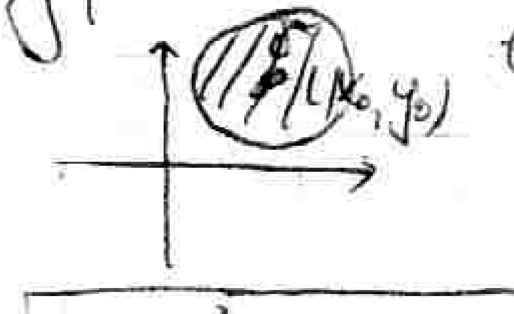
$\varphi(t) = \sin t$ има непр. пр. в/у $[0, \frac{\pi}{2}]$

2) $\int_1^2 x \ln x\, dx = \int_1^2 \ln x\, d(\frac{x^2}{2}) = \frac{1}{2} \int_1^2 \ln x\, dx^2 = \frac{1}{2} [x^2 \ln x \Big|_1^2 - \int_1^2 x^2 \ln x] =$

$= \frac{1}{2} [(4 \ln 2 - \ln 1) - \int_1^2 x^2 d \ln x] = \frac{1}{2} [(4 \ln 2 - \ln 1) - \int_1^2 x^2 \frac{1}{x} . dx] =$

$= \frac{1}{2} [4 \ln 2 - \overset{0}{\ln 1} - \frac{x^2}{2} \Big|_1^2] = \frac{1}{2} [4 \ln 2 - \frac{3}{2}] = 2 \ln 2 - \frac{3}{4}$

## 7. Лице на равнинна фигура

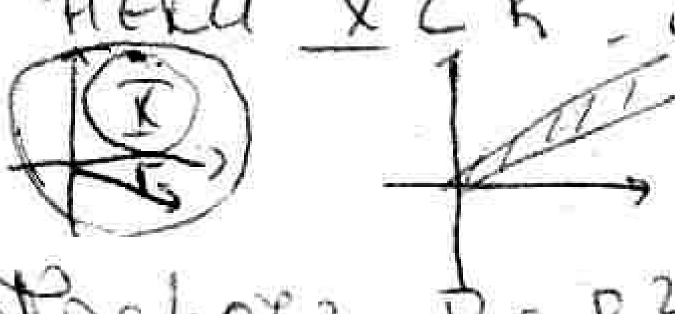


Деф $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$ $(x_0 - \delta, x_0 + \delta)$

Деф Нека $M_0(x_0, y_0) \in \mathbb{R}^2$, $\delta > 0$

$B_\delta(x_0, y_0) = B_\delta(M_0) = \{M \in \mathbb{R}^2 : |MM_0| < \delta\} = \{(x,y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}$

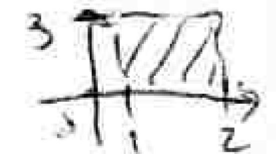
Деф Нека $\overline{X} \subset \mathbb{R}^2$. Казваме, че $\overline{X}$ е огр., ако $\exists$ $B_r(0,0) : \overline{X} \subset B_r(0,0)$



$\to$ не е огр.

Деф Правоъг. $P \in \mathbb{R}^2$ е всяко м-во от вида $P = \{(x,y) : a \alpha x \alpha b, c \alpha y \alpha d\}$, където $\alpha \leftrightarrow <, \leq$

Пример: $P = \{(x,y) : 1 \leq x < 2, 0 \leq y \leq 3\}$



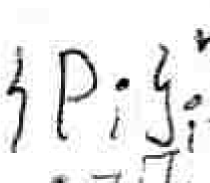
$\to$ не $\to$ успоредни на $\vec{Ox}$ и $\vec{Oy}$

Деф Нека $P = \{(x,y) : a \alpha x \alpha b, c \alpha y \alpha d\}$ е прав. Вътрешност на правоъгълник $P$, наричаме м-вото $P^\circ = \{(x,y) : a < x < b, c < y < d\}$

Деф Лице на правоъгълник $P = \{(x,y) : a < x < b, c < y < d\}$ наричаме числото $S(P) = (b-a)(d-c)$

Деф М-вото $K \subset \mathbb{R}^2$ се нарича клетъчно, ако $\exists$ $\{P_i\}_{i=1}^n$, $P_i$ – правоъг., $P_j^\circ \cap P_i^\circ = \emptyset$, $\forall i,j = 1 \div n$, $i \neq j$

$K = \bigcup_{i=1}^n P_i$



имат най-мн. общи гр.

Деф Нека $K$ – кл. м-во. $K = \bigcup_{i=1}^n P_i : P_i \cap P_j^\circ = \emptyset$ $\forall i,j = 1 \div n$, $i \neq j$. Лице на $K$ наричаме числото:

$S(K) = \sum_{i=1}^n S(P_i)$

Забележка: 1) Ако $\exists$ кл. м-во $K = \bigcup_{i=1}^{n} P_i = \bigcup_{s=1}^{m} Q_s$

$(P_i^\circ \cap P_j^\circ = \emptyset,\ Q_r^\circ \cap Q_l^\circ = \emptyset,\ r \neq l,\ i \neq j)$

$\Rightarrow \sum_{i=1}^{n} S(P_i) = \sum_{s=1}^{m} S(Q_s) = S(K)$

2) $K_1$ и $K_2$ - кл. м-ва : $K_1^\circ \cap K_2^\circ = \emptyset \Rightarrow$

$S(K_1 \cup K_2) = S(K_1) + S(K_2)$

? 3) Ако $K_1 \subset K_2 \Rightarrow S(K_1) \leq S(K_2)$

4) Ако $K \in$ клетъчно м-во и $x_0 \in \mathbb{R}^2 \Rightarrow S(x_0 + K) = S(K)$

$x_0 + K = \{x_0 + x : x \in K\}$

Деф| Нека $G \subset \mathbb{R}^2$. Казваме, че <u>м-вото $G \in$ измеримо</u>, ако за всяко $\varepsilon > 0$, $\exists$ кл. м-во $A$ и $B$:
1) $A \subset G \subset B$
2) $S(B) - S(A) < \varepsilon$

Деф| Нека м-вото $G \subset \mathbb{R}^2$ е измеримо
$\Rightarrow$ едно число $S(G)$ се нарича <u>лице на $G$</u>, ако $\forall$ кл. м-во $K$ : $K \subset G \subset \mathcal{K} \Rightarrow S(K) \leq S(G) \leq S(\mathcal{K})$

Т| $\forall$ измеримо м-во $E \subset \mathbb{R}^2$ има единствено лице.

<u>Д-во:</u>

Нека $E$ - изм. м-во в $\mathbb{R}^2$
$\forall$ клетъчно м-во $K, \mathcal{K}$ $K \subset E \subset \mathcal{K} \Rightarrow S(K) \leq S(\mathcal{K})$
$\Rightarrow \exists$ ~~sup~~ $S(K) \leq \sup_{K \subset E} S(K) \leq S(\mathcal{K}) \Rightarrow$

$\exists \inf_{E \subset \mathcal{K}} S(\mathcal{K}) \Rightarrow S(K) \leq \sup S(K) \leq \inf S(\mathcal{K}) \leq S(\mathcal{K})$ за $\forall K, \mathcal{K} : K \subset E \subset \mathcal{K}$

$\Rightarrow$ Ако $S(E)$ е лице на $E \Rightarrow S(E) \leq \inf_{E \subset \mathcal{K}} S(\mathcal{K})$

$$(+)\begin{cases} \inf_{E \subset \mathcal{K}} S(\mathcal{K}) \leq S(\mathcal{K}) \\ -\sup_{K \subset E} S(K) \leq -S(K) \end{cases}$$

$\inf_{E \subset \mathcal{K}} S(\mathcal{K}) - \sup_{K \subset E} S(K) \leq S(\mathcal{K}) - S(K) \leq S(\mathcal{K}_\varepsilon) - S(K_\varepsilon) < \varepsilon$

$E$ - измеримо м-во $\Rightarrow \forall \varepsilon > 0 \ \exists K, \mathcal{K}$: 1) $K_\varepsilon \subset E \subset \mathcal{K}_\varepsilon$
2) $S(\mathcal{K}_\varepsilon) - S(K_\varepsilon) < \varepsilon$

$\Rightarrow \inf S(\mathcal{K}) - \sup_{K \subset E} S(K) \cancel{\leq \varepsilon} = 0$

$\Rightarrow \underbrace{\inf_{E \subset \mathcal{K}} S(\mathcal{K}) = \sup_{K \subset E} S(K)}_{S(E)} \Rightarrow$ лицето е !

Т (Критерий за измеримост): Мн. $G \subset \mathbb{R}^2$ е измеримо $\Leftrightarrow$
$\forall \varepsilon > 0, \exists$ изм. м-ва $E, F$: 1) $E \subset G \subset F$
2) $S(F) - S(E) < \varepsilon$

<u>Д-во:</u>

Ако $G$ - измеримо $\overset{\text{Деф}}{\Rightarrow} \forall \varepsilon > 0, \exists$ кл. м-ва $K, \mathcal{K}$: 1) $K \subset G \subset \mathcal{K}$
2) $S(\mathcal{K}) - S(K) < \varepsilon$

$\Leftarrow$| Нека $\forall \varepsilon > 0, \exists$ изм. м-ва $E$ и $F$: 1) $E \subset G \subset F$
2) $S(F) - S(E) < \frac{\varepsilon}{3}$ (1)?

Т.к. $E \subset G$ е изм, то $\forall \frac{\varepsilon}{3} > 0, \exists$ кл. м-во $K \subset E$: $S(F) - S(K) < \frac{\varepsilon}{3}$ (
Т.к. $S(E) = \sup_{K \subset E} S(K)$

Т.к. $G \subset F$ изм м-во $\Rightarrow \forall \frac{\varepsilon}{3} > 0 \ \exists$ кл. м-во $\mathcal{K}_\varepsilon$: $F \subset \mathcal{K}_\varepsilon$ и $S(\mathcal{K}_\varepsilon) - S(F) < \frac{\varepsilon}{3}$ (

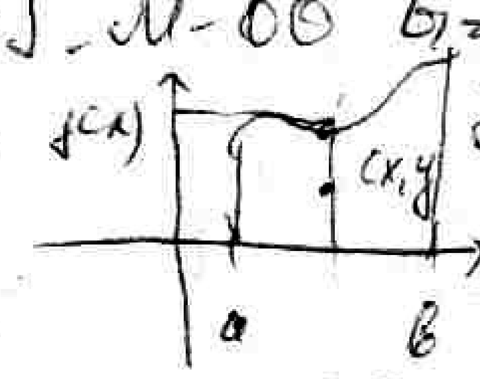
т.к. ~~G⊂F⊂K~~ $S(F)=\inf_{F\subset K} S(K)$ 1) $K_\varepsilon \subset E\subset G\subset F\subset K_\varepsilon$

2) $S(K_{\varepsilon'})-S(k_{\varepsilon'})=$

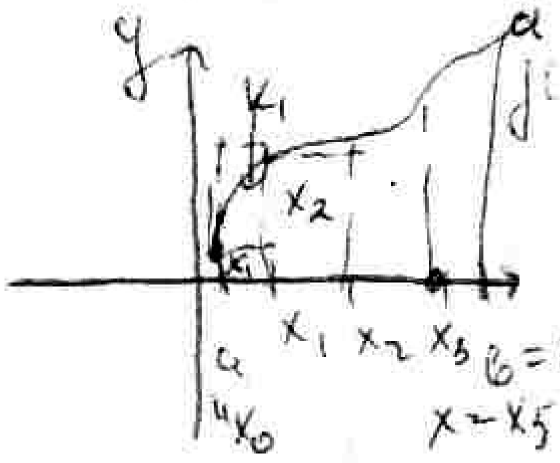
$=[S(K_{\varepsilon'})-S(F)]+[S(F)-S(F)]+[S(E)-S(k_{\varepsilon'})]\overset{1,2,3}{<}\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$

$\overset{\text{Деф}}{\Rightarrow}$ G е измеримо

Деф| Нека ф-та $f(x)\ge 0$ и непр. в/у $[a,b]$. М-во $G=\{(x,y): a\le x\le b, 0\le y\le f(x)\}$ се нарича криволинеен трапец




Т| Нека $f(x)\ge 0$ и непр. в/у $[a,b]$ => кривол. тр. $G=\{(x,y): a\le x\le b, 0\le y\le f(x)\}$ е изм. м-во в $R^2$ и $S(G)=\int_a^b f(x)\,dx$




Д-во:

Нека $\tau=\{x_i\}_{i=0}^n$ е разб. на $[a,b]$

$m_i=\inf_{x\in[x_{i-1},x_i]} f(x)$

$M_i=\sup_{x\in[x_{i-1},x_i]} f(x)$ } $\forall\, i=1\div n$

$k_i=[x_{i-1},x_i]\times[0,m_i]$

$K_i=[x_{i-1},x_i]\times[0,M_i]$

Нека $k=\bigcup_{i=1}^n k_i$, $K=\bigcup_{i=1}^n K_i$ => $k, K$ - кл. м-ва, $k\subset G\subset K$

$S(k)=\sum_{i=1}^n S(k_i)=\sum_{i=1}^n m_i\Delta x_i=s_\tau(f)$

$S(K)=\sum_{i=1}^n M_i\Delta x_i=S_\tau$

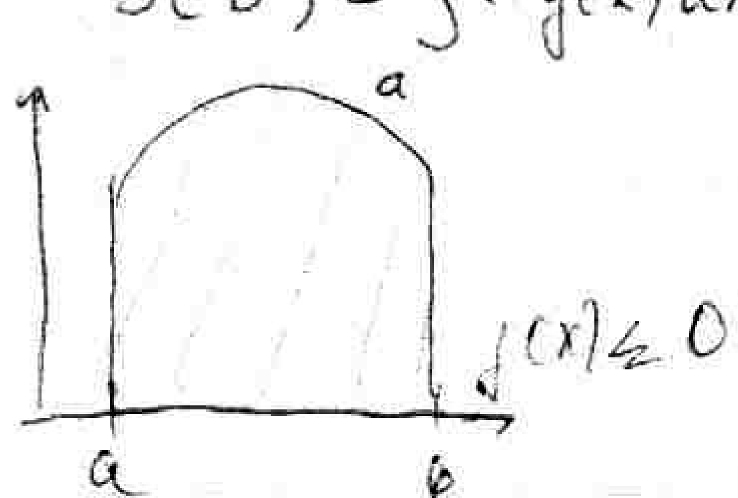
$f(x)$ е непр. в/у $[a,b]$ => $f(x)$ е инт. $\overset{\text{кр. инт.}}{\Rightarrow}$ $\forall\varepsilon>0, \exists\delta=\delta(\varepsilon)>0: \forall\tau=\{x_i\}_{i=0}^n$: $s_\tau<S_\tau$ => $S_\tau-s_\tau<\varepsilon$

=> ∃ кл. м-ва $k, K$: 1) $k\subset G\subset K$
2) $S(K)-S(k)=S_\tau-s_\tau<\varepsilon$

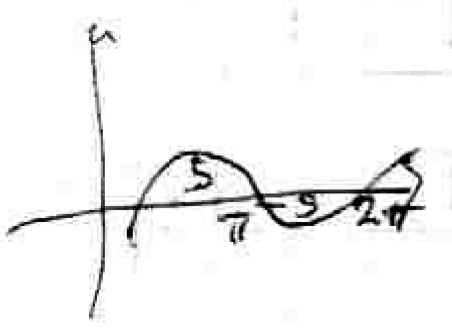
=> G е измеримо

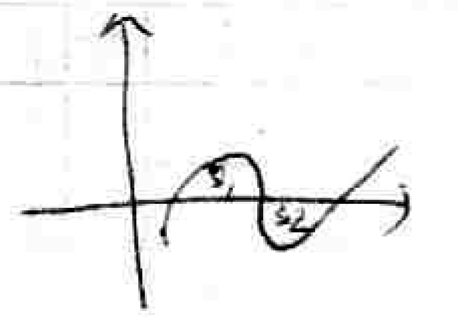
$S(G)=\sup_{k\subset G} S(k)=\sup s_\tau=\int_a^b f(x)\,dx$

Деф| $f(x), g(x)$ - непр. в/у $[a,b]$
$D=\{(x,y): a\le x\le b, f(x)\le y\le g(x)\}$, D - изм. м-во
$S(D)=\int_a^b g(x)\,dx-\int_a^b f(x)\,dx=\int_a^b[g(x)-f(x)]\,dx$



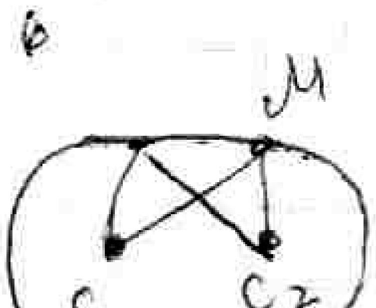

$\int_a^b f(x)\,dx=-S(G)$



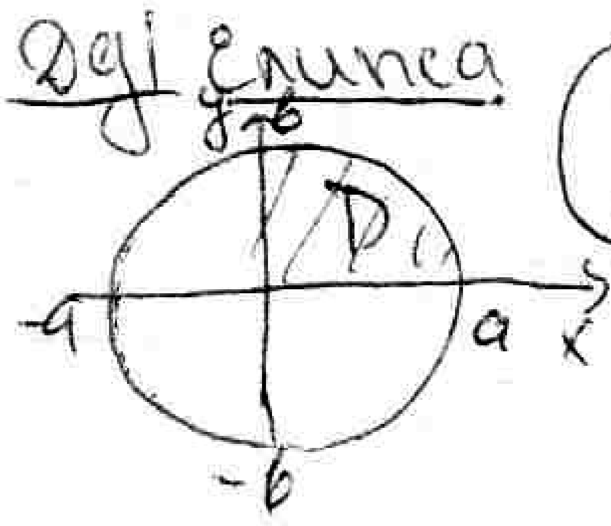


$S_1-S_2$

Деф| елипса



$\mathcal{E}=\{M: |MC_1|+|MC_2|=const\}$



$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ уравнение на елипса

=> Ако $(x,y)\in\mathcal{E}$ => $(-x,y),(x,-y),(-x,-y)\in\mathcal{E}$

$S(\mathcal{E})=4S(D)$,

$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$, но $y\ge 0$

$D=\{(x,y): 0\le x\le a, 0\le y\le b\sqrt{1-\frac{x^2}{a^2}}\}$ =>

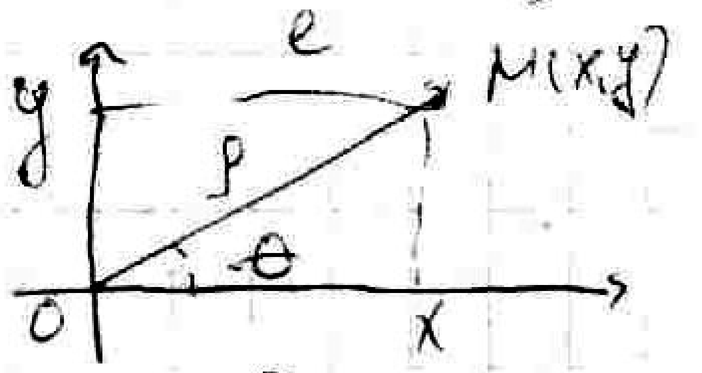
$S(D)=\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=ab\int_0^1\sqrt{1-\left(\frac{x}{a}\right)^2}\,d\left(\frac{x}{a}\right)=$ $t=\frac{x}{a}$

$$= ab\int_0^1 \underbrace{\sqrt{1-t^2}}_{\pi/4}\,dt = \frac{\pi ab}{4} \Rightarrow S(E) = 4S(D) = \pi ab \Rightarrow$$

$$\boxed{S(E) = \pi ab}$$

• Ако $a = b = R \Rightarrow S_{\text{кръг}} = \pi R^2$

**Деф** Полярна координатна с-ма $M(\theta, \rho)$, $\rho(0,0)$ $\Big|$ $0 \le \theta < 2\pi$, $0 < \rho$

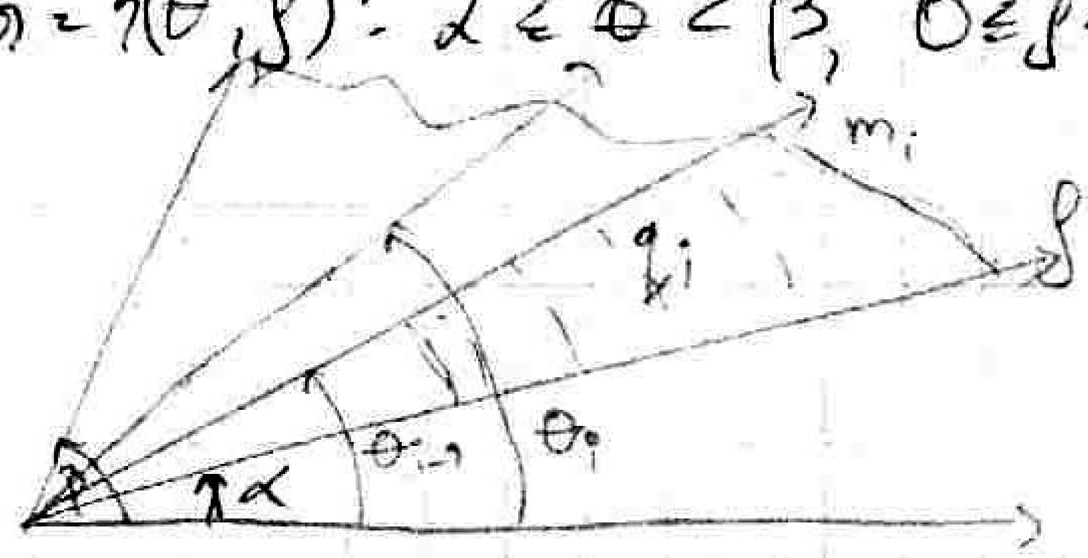

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ } връзка м/у дек. и пол. к. с-ма

**Деф** Нека $\rho : \rho(\theta)$ – непр. ф-я $[\alpha, \beta] \subset [0, 2\pi]$

$G = \{(\theta, \rho) : \alpha \le \theta \le \beta,\ 0 \le \rho \le \rho(\theta)\}$ → криволинеен сектор

**Т** Нека ф-та $\rho = \rho(\theta)$ е непр. в/у $[\alpha, \beta]$. Тогава кр. сектор

$G = \{(\theta, \rho) : \alpha \le \theta \le \beta,\ 0 \le \rho \le \rho(\theta)\}$ е изм. м-во и $S(G) = \frac{1}{2}\int_\alpha^\beta \rho^2(\theta)\,d\theta$




**Д-во:** Нека $\tau = \{\theta_i\}_{i=0}^n$ – разб. на $[\alpha, \beta]$

$\alpha = \theta_0 < \theta_1 < \dots < \theta_{n-1} < \theta_n = \beta$

$[\theta_{i-1}, \theta_i] \to \Delta\theta_i = \theta_i - \theta_{i-1}$

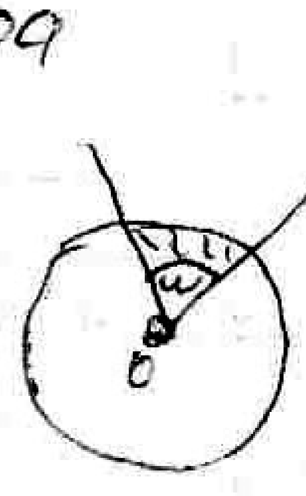
$m_i = \inf\limits_{\theta \in [\theta_{i-1}, \theta_i]} \rho(\theta)$, $M_i = \sup\limits_{\theta \in [\theta_{i-1}, \theta_i]} \rho(\theta)$

$q_i = \{(\theta, \rho) : \theta_{i-1} \le \theta \le \theta_i,\ 0 \le \rho \le m_i\}$ → кръгов ъглов сектор } изм. м-ва

$Q_i = \{(\theta, \rho) : \theta_{i-1} \le \theta \le \theta_i,\ 0 \le \rho \le M_i\}$

Нека $q = \bigcup\limits_{i=1}^n q_i$ } измерими $q \subset G \subset Q$

$Q = \bigcup\limits_{i=1}^n Q_i$




$\frac{\pi r^2 \omega}{2\pi} = \frac{1}{2} r^2 \omega$

$S(q) = \sum\limits_{i=1}^n S(q_i) = \sum\limits_{i=1}^n \frac{1}{2} m_i^2 \Delta\theta_i$ $\Rightarrow$ $S(q) \to s_\tau(\frac{1}{2}\rho^2(\theta))$

$S(Q) = \sum\limits_{i=1}^n S(Q_i) = \sum\limits_{i=1}^n \frac{1}{2} M_i^2 \Delta\theta_i$ $\quad S(Q) \to S_\tau(\frac{1}{2}\rho^2(\theta))$

$\Rightarrow \rho(\theta) = \frac{1}{2}\rho^2(\theta)$ е непр. в/у $[a, b]$ → интегруема в/у $[\alpha, \beta] \Rightarrow$

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{\theta_i\}_{i=0}^n, \delta\tau < \delta \Rightarrow$

$$\underbrace{S_\tau(\tfrac{1}{2}\rho^2) - s_\tau(\tfrac{1}{2}\rho^2)}_{\Delta} < \varepsilon, \text{ но}$$

1) $\Delta = S(Q) - S(q) < \varepsilon$
2) $q \subset G \subset Q$ } $\Rightarrow G$ е измеримо м-во

$S(G) = \sup\limits_{q \subset G} S(q) = \sup s_\tau(\frac{1}{2}\rho^2) = \int_\alpha^\beta \frac{1}{2}\rho^2(\theta)\,d\theta = \frac{1}{2}\int_\alpha^\beta \rho^2(\theta)\,d\theta$

**Деф** Окр. (R): $\rho = \rho(\theta) = R,\ 0 \le \theta \le 2\pi$

$B_R(\theta): \{(\theta, \rho): 0 \le \theta \le 2\pi,\ 0 \le \rho \le R\}$

$S(B_R(0)) = \frac{1}{2}\int_0^{2\pi} R^2\, d\theta = \frac{1}{2}R^2 \cdot \int_0^{2\pi} 1\, d\theta = R^2 2\pi \cdot \frac{1}{2} = \pi R^2$

- $\rho = a(1+\cos\theta),\ 0 \le \theta \le 2\pi$

$$S(G) = \frac{1}{2}\int_0^{2\pi} a^2(1+\cos\theta)^2 d\theta = \frac{a^2}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\, d\theta =$$

$$= \frac{a^2}{2}\left[\int_0^{2\pi} 1\, d\theta + 2\int_0^{2\pi}\cos\theta\, d\theta + \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\, d\theta\right] =$$

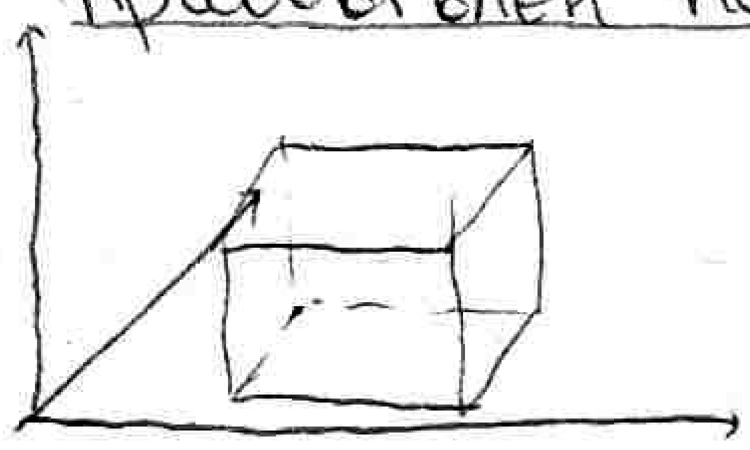
$$= \frac{a^2}{2}\left[2\pi + 2\sin\theta\Big|_0^{2\pi} + \frac{1}{2}\left(2\pi + \frac{1}{2}\sin 2\theta\Big|_0^{2\pi}\right)\right] =$$

$$= \frac{a^2}{2}(2\pi + \pi) = \frac{3}{2}\pi a^2$$

## 8. Обем на тяло с известно напречно сечение. Обем на ротационно тяло

**Деф** $\Pi = \langle a_1, b_1\rangle \times \langle a_2, b_2\rangle \times \langle a_3, b_3\rangle$, където $a_i, b_i \in \mathbb{R}$, $i = 1,2,3$, $\langle \in \{(, [\}$, $\rangle \in \{), ]\}$

правоъгълен паралелепипед



Вътрешност на $\Pi$: $\Pi^\circ = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$

**Деф** Клетъчно тяло наричаме $\forall$ мн. $K = \bigcup_{i=1}^{n} \Pi_i$:

$\Pi_i^\circ \cap \Pi_j^\circ = \emptyset\ (\forall i, j = 1 \div n,\ i \ne j)$

**Деф** $V(\Pi) = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$ – обем на $\Pi$

$V(K) = \sum_{i=1}^{n} V(\Pi_i)$ – обем на кл. тяло

**Деф** Нека $\Omega \subset \mathbb{R}^3$. Казваме, че $\Omega$ е <u>измеримо м-во/тяло</u>, ако за $\forall \varepsilon > 0$, $\exists$ кл. тяло $k, K$: 1) $k \subset \Omega \subset K$
2) $V(K) - V(k) < \varepsilon$

**Деф** Нека $\Omega$ е измеримо тяло. <u>Обем на $\Omega$</u> се нарича такова число $V(\Omega)$: $\forall k, K$ (кл. тела): $k \subset \Omega \subset K \Rightarrow$
$V(k) \le V(\Omega) \le V(K)$

**Т** Ако $\Omega$ е измеримо тяло в $\mathbb{R}^3$, то $\exists!\ V(\Omega)$ на $\Omega$. При това

$$V(\Omega) = \sup_{k \subset \Omega} V(k) = \inf_{\Omega \subset K} V(K)$$

Д-во:

Нека $\Omega$ – измеримо м-во в $\mathbb{R}^3$

$\forall$ клетъчно м-во $k, K$: $k \subset \Omega \subset K \Rightarrow V(k) \le V(K)$

$\Rightarrow \exists\ V(k) \le \sup_{k \subset \Omega} V(k) \le V(K)$

$\Rightarrow \exists \inf_{\Omega \subset K} V(K) \Rightarrow V(k) \le \sup V(k) \le \inf V(K) \le V(K)$ за $\forall k, K$ $k \subset \Omega \subset K$

$\Rightarrow$ ако $V(\Omega)$ е обем на $\Omega \Rightarrow V(\Omega) \le \inf_{\Omega \subset K} V(K)$

$+\begin{cases} \inf_{\Omega \subset K} V(k) \le V(K) \\ -\sup V(k) \le -V(K) \end{cases}$

$\Rightarrow \inf_{\Omega\subset\mathcal{K}} V(\mathcal{K}) - \sup_{K\subset\Omega} V(K) \le V(\mathcal{K}) - V(K) \le V(\mathcal{K}_\varepsilon) - V(K_\varepsilon) < \varepsilon$

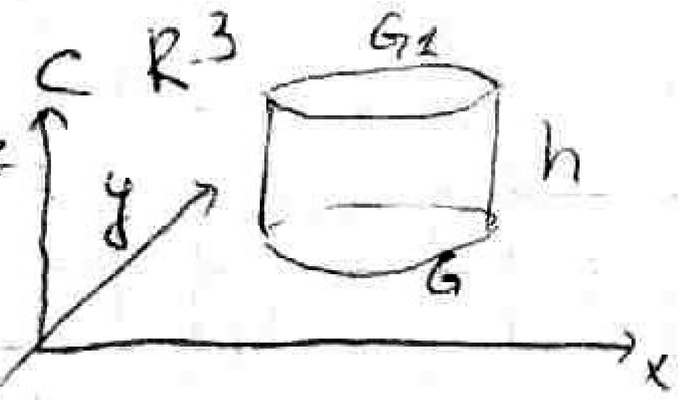
$\Omega$ - измеримо м-во $\Rightarrow \forall \varepsilon>0: \exists K, \mathcal{K}:$ 1) $K_\varepsilon \subset \Omega \subset \mathcal{K}_\varepsilon$
2) $V(\mathcal{K}_\varepsilon) - V(K_\varepsilon) < \varepsilon$

$\Rightarrow \inf_{\Omega\subset\mathcal{K}} V(\mathcal{K}) - \sup_{K\subset\Omega} V(K) = 0$

$\Rightarrow \underbrace{\inf_{\Omega\subset\mathcal{K}} V(\mathcal{K}) = \sup_{K\subset\Omega} V(K)}_{V(\Omega)} \Rightarrow$ лицето $\in$ !

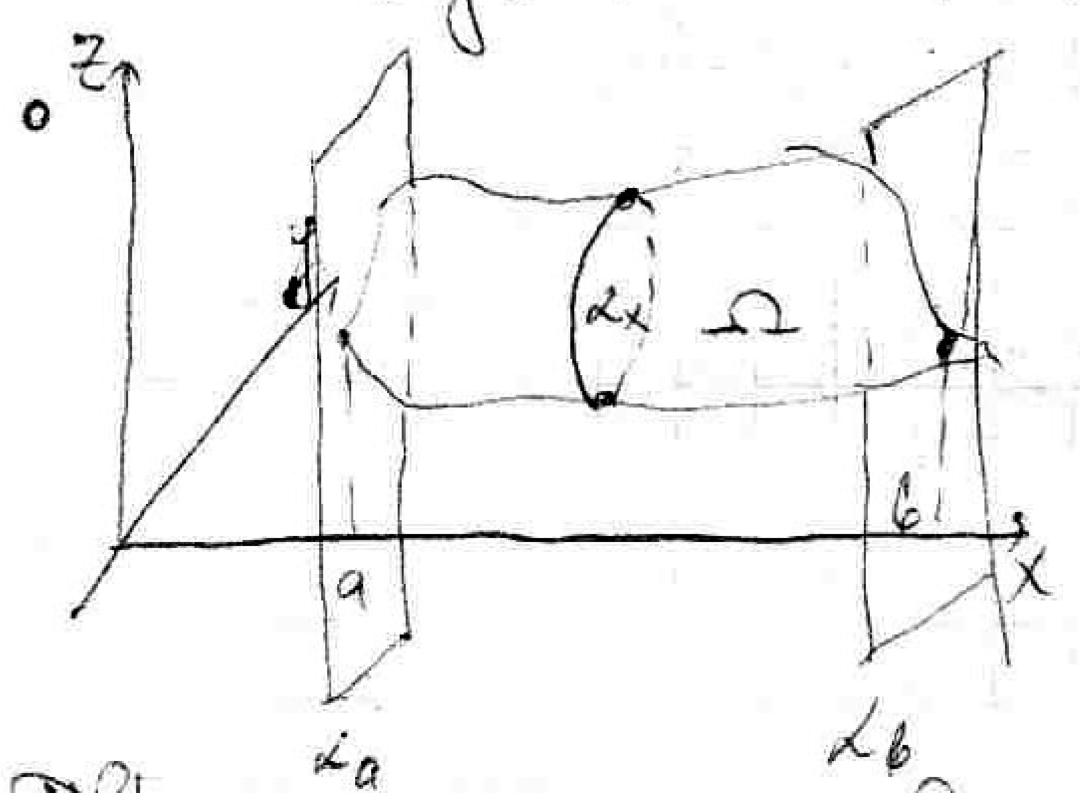
**Т)** Тялото $\Omega$ е измеримо $\Leftrightarrow$ за $\forall \varepsilon>0, \exists$ изм. тела: $E, F$:
1) $E \subset \Omega \subset F$
2) $V(F) - V(E) < \varepsilon$

**Деф)** Нека $G\subset R^2$, $h>0$
Множ. $Ц(G) = \{(x,y,z) : (x,y)\in G, 0\le z\le h\} \subset R^3$
се нарича цилиндър с осн. $G$
$G = \{(x,y,z) : (x,y)\in G, z=0\}$ } осн. на
$G_1 = \{(x,y,z) : (x,y)\in G, z=h\}$ } цилиндъра




**Т6)** Нека $G\subset R^2$ е измеримо мн, $h>0$
$\Rightarrow Ц(G) = \{(x,y,z) : (x,y)\in G, 0\le z\le h\}$ е изм. м-во и обема
$\boxed{V(Ц(G)) = S(G).h}$

Д-во:
① $G$ - изм. м-во $\Rightarrow \forall \varepsilon>0, \exists$ кл. мн-ва $K$ и $\mathcal{K}$:
1) $K\subset G\subset \mathcal{K}$
2) $S(\mathcal{K}) - S(K) < \varepsilon/h$
$\Pi(K) = K\times[0,h]$ - кл. тяло в $R^3$
$\Pi(\mathcal{K}) = \mathcal{K}\times[0,h]$ - кл. тяло в $R^3$
т.к. $K\subset G\subset\mathcal{K} \Rightarrow \underbrace{K\times[0,h]}_{\Pi(K)} \subset \underbrace{G\times[0,h]}_{Ц(G)} \subset \underbrace{\mathcal{K}\times[0,h]}_{\Pi(\mathcal{K})}$ - кл. тела
$V(\Pi(\mathcal{K}) - \Pi(K)) = (S(\mathcal{K}) - S(K))\, h < \frac{\varepsilon}{h}.h$
$\Rightarrow Ц(G)$ е измеримо м-во
② $V(Ц(G)) = \sup_{P\subset Ц(G)} V(P) = \sup_{K\subset G} S(K).h = h.\sup_{K\subset G} S(K) = h.S(G)$
$P = K\times[0,h]$ - $K$ - кл. м-во, $K\in R^2$
$\Rightarrow V(Ц(G)) = S(G).h$

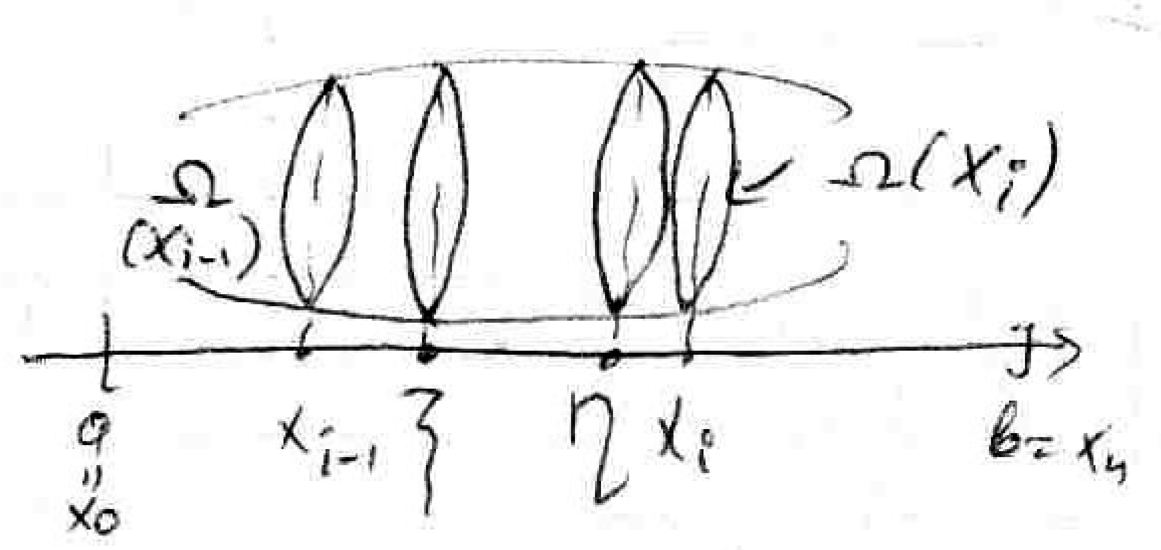



$x\in Ox \Rightarrow \alpha_x \ni x : \alpha_x \perp Ox$
$(\alpha_x \parallel yOz)$
$\Omega\subset R^3$, $\forall (x,y,z)\in\Omega: a\le x\le b$
$\Rightarrow \forall x\in[a,b] \Rightarrow \alpha_x \cap \Omega \ne \emptyset$
$\Omega(x) = \alpha_x \cap \Omega$
$\forall x\in[a,b]: \Omega(x)$ е измеримо и $S(x) =$ лице$(\Omega(x))$
$S: [a,b]\to R$
$S = S(x)$ - непр.
$\forall x,y\in[a,b]: Пр_{yOz}\,\Omega(x) \subseteq Пр_{yOz}\,\Omega(y)$

**Т)** При горните предп. имаме: 1) $\Omega$ е изм. 2) $V(\Omega) = \int_a^b S(x)\,dx$

Д-во:

$\tau = \{x_i\}_{i=0}^n$ - разб. на $[a,b]$
$m_i = \inf_{x\in[x_{i-1},x_i]} S(x)$, $M_i = \sup_{x\in[x_{i-1},x_i]} S(x) \Rightarrow$
$\exists \xi_i \in [x_{i-1},x_i] : S(\xi_i) = m_i$
$\exists \eta \in [x_{i-1},x_i] : S(\eta_i) = M_i$ $\}\ \forall i = 1\div n$



$D_i = \Omega(\xi_i) \Rightarrow Ц(D_i) = D_i \times [x_{i-1}, x_i]$
$P_i = \Omega(\eta_i) \Rightarrow Ц(P_i) = P_i \times [x_{i-1}, x_i]$

$\bigcup_{i=1}^{n} Ц^{изм}(D_i) \subset \Omega \subseteq \bigcup_{i=1}^{n} Ц^{изм}(D_i)$

$V(\bigcup_{i=1}^{n} Ц(D_i)) = \sum_{i=1}^{n} V(Ц(D_i)) = \sum_{i=1}^{n} лиц(D_i)(x_{i-1}, x_i) = \sum_{i=1}^{n} S(\xi_i)\Delta x_i = \sum_{i=1}^{n} m_i \Delta x_i = S_\tau(S(x))$

$V(\bigcup_{i=1}^{n} Ц(P_i)) = \sum_{i=1}^{n} V(Ц(P_i)) = \sum_{i=1}^{n} лиц(P_i)(x_{i-1}, x_i) = \sum_{i=1}^{n} S(\eta_i)\Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i = \mathcal{S}_\tau(S(x))$

$S(x)$ – непр. в/у $[a,b] \Rightarrow S(x)$ – инт. $\Rightarrow$

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=0}^{n}, \delta_\tau < \delta \Rightarrow \mathcal{S}_\tau(S(x)) - S_\tau(S(x)) < \varepsilon$

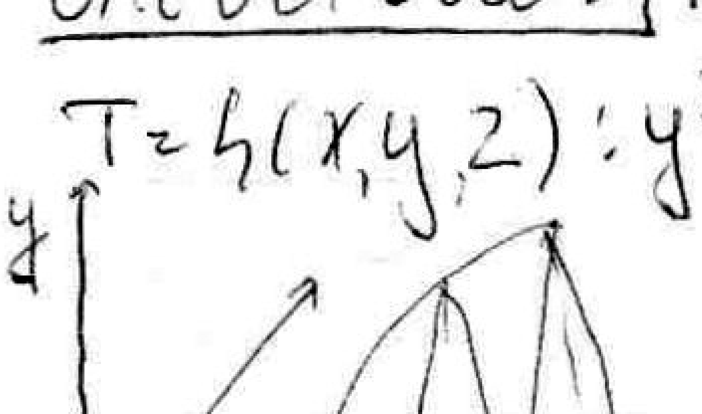
$\Rightarrow V(\bigcup_{i=1}^{n} Ц(P_i)) - V(\bigcup_{i=1}^{n} Ц(D_i)) = \mathcal{S}(S(x)) - S(S(x)) < \varepsilon$

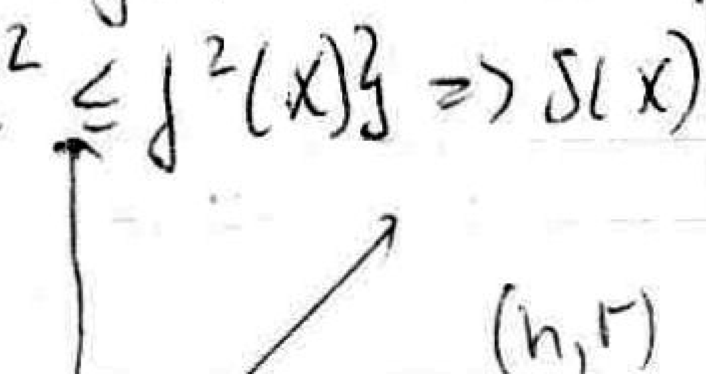
$\Rightarrow \Omega$ – изм. тяло в $\mathbb{R}^3$

$V(\Omega) = \sup_\tau V(\bigcup_{i=1}^{n} Ц(D_i)) = \sup_\tau V(\bigcup_{i=1}^{n} Ц(P_i)) = \int_a^b S(x)\,dx$

<u>Следствие:</u> Нека $f(x) \geq 0$ и непр. в/у $[a,b]$

$T = \{(x,y,z) : y^2 + z^2 \leq f^2(x)\} \Rightarrow S(x) = \pi f^2(x)$ и $V(T) = \int_a^b \pi f^2(x)\,dx = \pi \int_a^b f^2(x)$

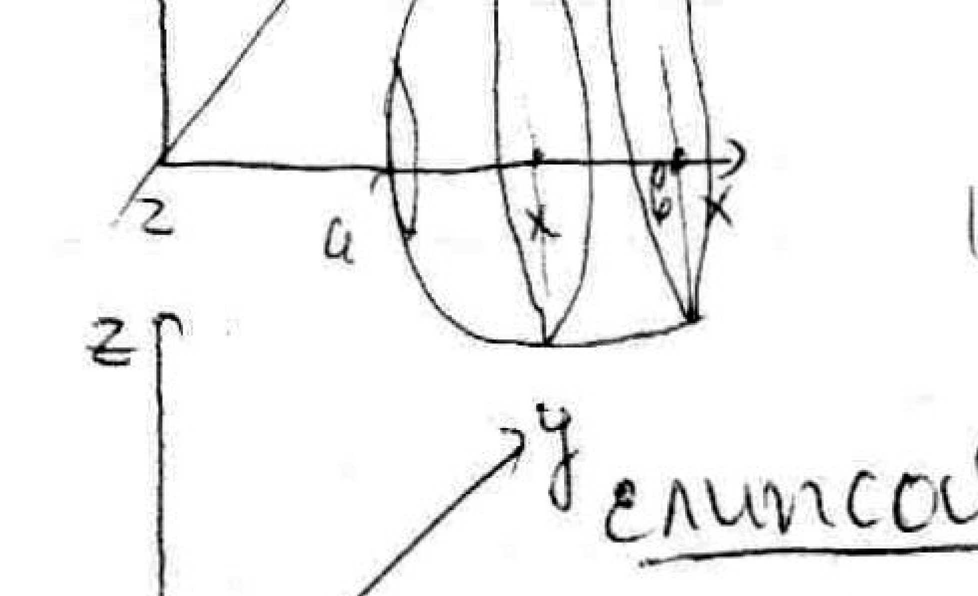




$l : y = \frac{r}{h} x$

$\Rightarrow V(конус) = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx =$

$= \frac{\pi r^2}{h^2} \int_0^h x^2\,dx = \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3}\Big|_0^h = \frac{\pi r^2}{\cancel{h^2}} \cdot \frac{h^{\cancel{3}}}{3} = \frac{B}{3}$



<u>елипсоид:</u>

$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$x \in [-a, a]$

$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} \Big| : \left(1 - \frac{x^2}{a^2}\right)$

$\mathcal{E}_x : \frac{y^2}{(b\sqrt{1-\frac{x^2}{a^2}})^2} + \frac{z^2}{(c\sqrt{1-\frac{x^2}{a^2}})^2} = 1$

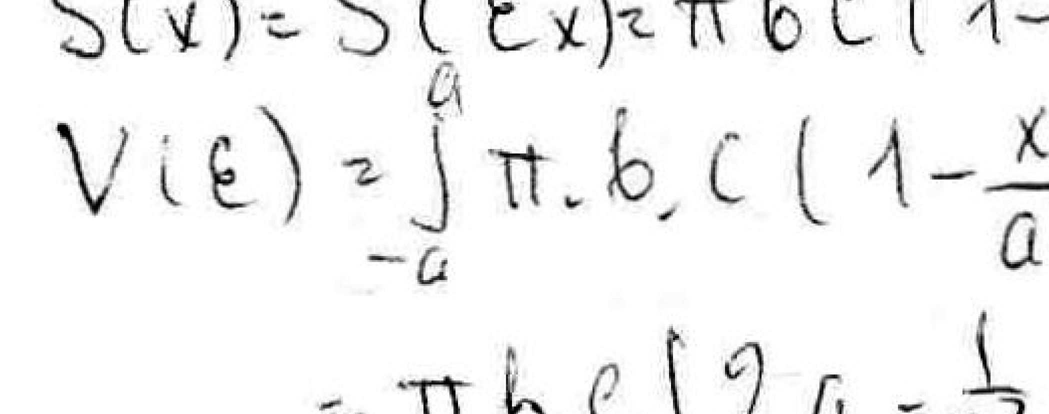
$S(x) = S(\mathcal{E}_x) = \pi b c \left(1 - \frac{x^2}{a^2}\right)$

$V(\mathcal{E}) = \int_{-a}^{a} \pi . b . c \left(1 - \frac{x^2}{a^2}\right) dx = \pi b c \left[\int_{-a}^{a} 1\,dx - \frac{1}{a^2}\int_{-a}^{a} x^2\,dx\right] =$

$= \pi b . c \left[2a - \frac{1}{a^2} \cdot \frac{x^3}{3}\Big|_{-a}^{a}\right] = \pi b c \left[2a - \frac{1}{\cancel{a^2}} \cdot \frac{2}{3} a^{\cancel{3}}\right] = \pi b c . \frac{4}{3} a$

$V_{кълбо} = \frac{4}{3}\pi . R^3 \quad (a = b = c = R)$

## 9. Дължина на крива линия.



$\tau : \{M_i\}_{i=0}^{n} \quad M_i \in l$

$M_0 = A, \ M_n = B$

$L(M_0 \ldots M_n) = \bigcup_{i=1}^{n} [M_{i-1}, M_i]$ – вписана начупена линия в $l$

$\Delta_\tau = \max |M_{i-1}, M_i|$ – ген. на разб.

$\delta_\tau = \max \Delta x_i, \ i = 1 \div n$

$0 \leq \delta_\tau \leq \Delta_\tau$

$d(L) =$ дълж. на $(L(M_0 \ldots M_n)) = \sum_{i=1}^{n} |M_{i-1}, M_i|$

Деф Дължина на кр. л. $\ell$, наричаме число $d(\ell)$:

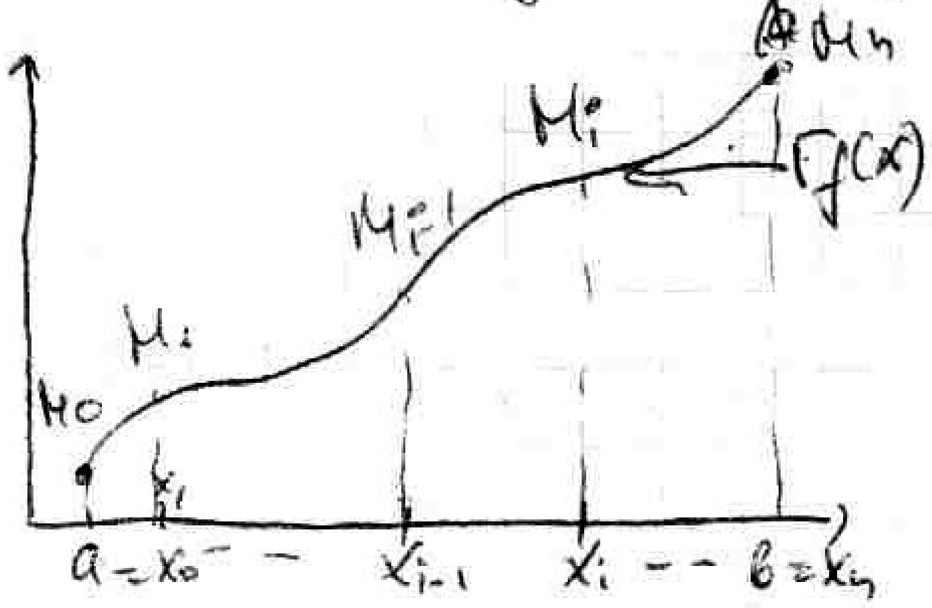
$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{M_i\}_{i=0}^{n}, \ \Delta_\tau < \delta$$

$$\Rightarrow |d(\ell) - d(L(M_0 \ldots M_n))| < \varepsilon$$

т.е. $d(\ell) = \lim\limits_{\delta \to 0} d(L(M_0 \ldots M_n))$

Т1 Нека $f(x)$ е диф в/у $[a,b]$ и $f'(x)$ е непр в/у $[a,b]$

$$\Rightarrow d(\Gamma f(x)) = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$




Д-во:

Нека $T = \{M_i\}_{i=0}^{n}$ на $\Gamma f(x)$, $i = 0 \div n$

Нека $M\{(x_i, f(x_i))\}_{i=0}^{n} \Rightarrow \tau = \{x_i\}_{i=0}^{n}$ р. на $[a,b]$

$L(M_0 \ldots M_n) \Rightarrow d(L(M_0 \ldots M_n)) = \sum\limits_{i=1}^{n} |M_{i-1}, M_i|$

$$|M_{i-1}, M_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \ (*) \ \forall i = 1 \div n$$

Разгл. $f$ в/у $[x_{i-1}, x_i]$ : $\overset{\text{Т.Л.}}{\Rightarrow} \exists \xi_i \in [x_{i-1}, x_i] : f(x_i) - f(x_{i-1}) = f'(\xi_i).(x_i - x_{i-1})$

$$\overset{(*)}{=} \sqrt{(x_i - x_{i-1})^2 + (f'(\xi_i))^2 (x_i - x_{i-1})^2} = \sqrt{1 + (f'(\xi_i))^2 . (\Delta x_i)^2} \Rightarrow$$

$\sum\limits_{i=1}^{n} \sqrt{1+[f'(\xi_i)]^2} . \Delta x_i$, където $\xi = \{\xi_i\}_{i=1}^{n} : \xi_i \in [x_{i-1}, x_i]$ за $\forall i = 1 \div n$

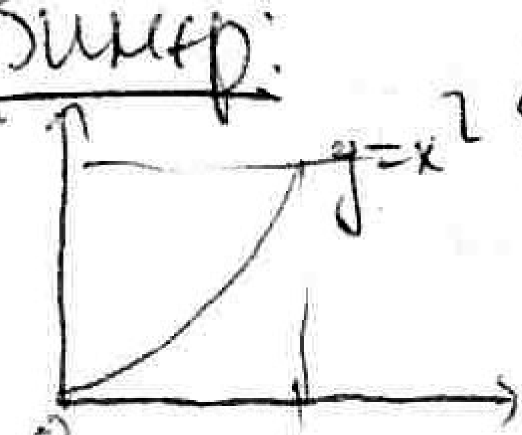
$\sigma_\tau(\sqrt{1+[f'(x)]^2}; \xi)$

$$\lim d(L(M_0 \ldots M_n)) = \lim_{\Delta_\tau \to 0} \sum_{i=1}^{n} \sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i = \lim_{\delta_\tau \to 0} \sum_{i=1}^{n} \sqrt{1+[f'(x_0)]^2}\,\Delta x_i =$$

$$= \lim_{\delta_\tau \to 0} \sigma_\tau(\sqrt{1+[f'(x)]^2}; \xi) = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$\underset{0}{\overset{0}{\downarrow}} \le \underset{0}{\overset{\delta_\tau}{\downarrow}} < \underset{0}{\overset{\Delta_\tau}{\downarrow}}$$

Пример:




$f(x) = x^2$, $[0,1]$

$y = x^2$, $d(\Gamma x^2) = \int_0^1 \sqrt{1+[(x^2)']^2}\,dx = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+(2x)^2}\,dx =$

$$= \frac{1}{2} \int \sqrt{1+(2x)^2}\,d(2x) \overset{2x=t}{=} \frac{1}{2} \int_0^2 \sqrt{1+t^2}\,dt$$

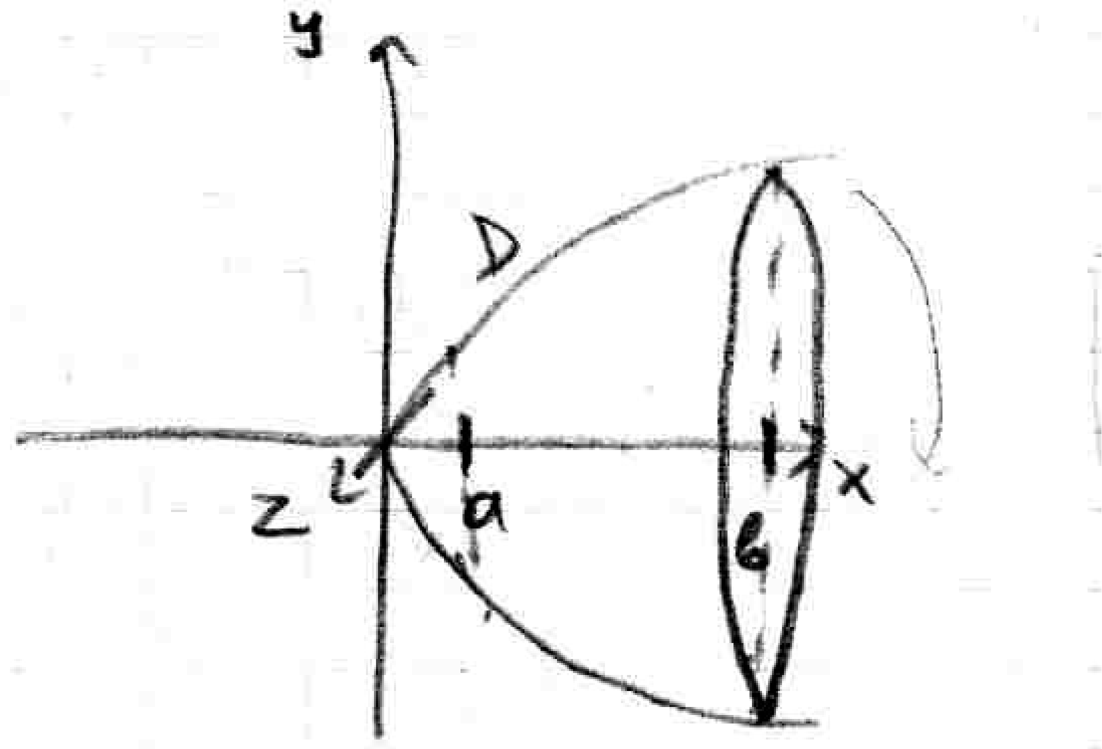
$$I = \int \sqrt{1+t^2}\,dt = t\sqrt{1+t^2} - \int t\,d\sqrt{1+t^2} = t\sqrt{1+t^2} - \int \frac{t^2+1-1}{\sqrt{1+t^2}}\,dt$$

$$= t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + \int \frac{1}{\sqrt{1+t^2}}\,dt = t\sqrt{1+t^2} - I + \ln(t + \sqrt{1+t^2}) + C$$

$$I = \frac{1}{2}\left(t\sqrt{1+t^2} + \ln(t + \sqrt{1+t^2})\right) + C$$

$$\frac{1}{2} \cdot \frac{1}{2}\left(t\sqrt{1+t^2} + \ln(t+\sqrt{1+t^2})\right)\Big|_0^2 = \frac{\sqrt{5}}{2} \ln \sqrt[4]{2+\sqrt{5}}$$

# (10) Лице на ротационна повърхнина




**Def** Нека $f(x)$ - непр. ф-я в/у $[a,b] \to \mathcal{P}(f)$

Нека $\tau = \{x_i\}_{i=0}^{n}$ - разб. на $[a,b]$, $\delta_\tau = \max\limits_{1 \le i \le n} \Delta x_i$

$M_i(x_i, f(x_i))$ $i = 0 \div n$

$L_\tau = M_0 M_1 \dots M_n$ - начупена линия

$\Rightarrow \mathcal{P}(L_\tau) = \bigcup\limits_{i=1}^{n} K_i$ - ротац. повърх., обр. от $L_\tau$,

$K_i$ - пресечен конус

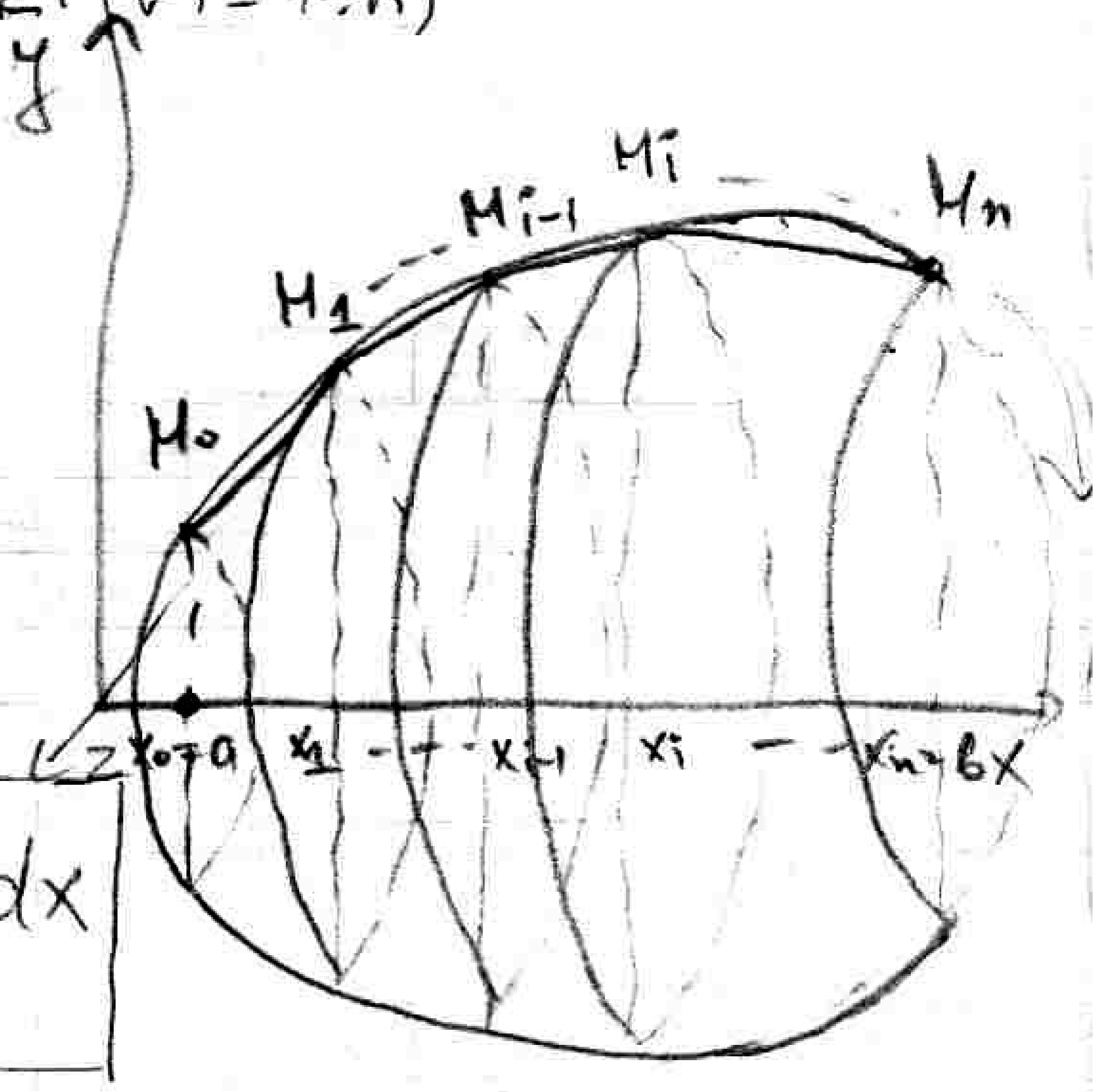
$S(\mathcal{P}(L_\tau)) = \sum\limits_{i=1}^{n} S(K_i)$, $S(K_i)$ - ок. повърх. на $K_i$ $(\forall i = 1 \div n)$

Лице $S(\mathcal{P}(f))$ е такова число, че:

$S(\mathcal{P}(f)) = \lim\limits_{\delta_\tau \to 0} S(\mathcal{P}(L_\tau))$, т.е.

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \tau = \{x_i\}_{i=0}^{n}, \delta_\tau < \delta$

$\Rightarrow |S(\mathcal{P}(f)) - S(\mathcal{P}(L_\tau))| < \varepsilon$




**Т** Ако $f(x) \ge 0$ и има непр. пр. $f'(x)$

в/у $[a,b] \Rightarrow$ $\boxed{S(\mathcal{P}(f)) = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx}$

## Д-во:

- Нека $\tau = \{x_i\}_0^n \to \mathcal{P}(L_\tau) = \bigcup\limits_{1}^{n} K_i$

$S(\mathcal{P}(L_\tau)) = \sum\limits_{i=1}^{n} S(K_i) = \sum\limits_{i=1}^{n} \pi(f(x_{i-1}) + f(x_i)) \cdot |M_{i-1}M_i| =$

$= \pi \sum\limits_{i=1}^{n} [f(x_{i-1}) - f(x_i)] \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$ (1)

т. Лагр. $\Rightarrow$ $\forall i = 1 \div n$, $\exists \xi_i \in (x_{i-1}, x_i) : f(x_i) - f(x_{i-1}) = f'(\xi_i) \cdot (\overbrace{x_i - x_{i-1}}^{\Delta x_i})$ (1)

$(1) = \pi \sum\limits_{i=1}^{n} [f(x_{i-1}) + f(x_i)] \sqrt{1 + [f'(\xi_i)]^2}\, \Delta x_i$

- Да разгл. ф-ята $F(x) = 2\pi f(x)\sqrt{1+[f'(x)]^2}\,\Delta x_i$ непр. в/у $[a,b] \Rightarrow$

$\exists I = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

$\Rightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0, \forall \tau = \{x_i\}_{i=0}^{n}, \delta_\tau < \delta, \delta = \min(\delta', \delta_1), \forall \xi = \{\xi_i\}_{i=1}^{n}$

$\Rightarrow |\sigma_\tau(F;\xi) - I| < \varepsilon$, т.е.

$\lim\limits_{\delta_\tau \to 0} \sigma_\tau(F;\xi) = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

- $\tau \to \mathcal{P}(L_\tau) \to S(\mathcal{P}(L_\tau)) = \pi \sum\limits_{i=1}^{n} [f(x_{i-1}) + f(x_i)] \sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$,

където $\xi = \{\xi_i\}_{i=1}^{n}$, $\xi_i \in (x_{i-1}, x_i)$ $\forall i = 1 \div n$

- $|I - S(\mathcal{P}(L_\tau))| = |[I - \sigma_\tau(F;\xi)] + [\sigma_\tau(F;\xi) - S(\mathcal{P}(L_\tau))]| \le$

$\le |I - \sigma_\tau(F;\xi)| + |\sigma_\tau(F;\xi) - S(\mathcal{P}(L_\tau))| <$

$< \varepsilon + |\sigma_\tau(F;\xi) - S(\mathcal{P}(L_\tau))|$ ★

• $|\sigma_\tau(f,\xi) - S(P(L_\tau))| = \left|\sum_{i=1}^{n} 2\pi f(\xi_i)\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i - \pi\sum_{i=1}^{n}[f(x_{i-1})+f(x_i)]\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i\right|$

$= \pi\left|\sum_{i=1}^{n}\left([\underbrace{f(\xi_i)-f(x_{i-1})}_{(2)\Rightarrow<\varepsilon}] + [\underbrace{f(\xi_i)-f(x_i)}_{(2)\Rightarrow<\varepsilon}]\right)\right|\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$

$\le \pi\sum_{i=1}^{n}\left(|f(\xi_i)-f(x_{i-1})| + |f(\xi_i)-f(x_i)|\right)\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i$

• $f(x)$ е непр. в/у $[a,b] \to f(x)$ е равном. непр $\Rightarrow$

$\varepsilon>0,\ \exists\ \delta' = \delta'(\varepsilon)>0:\ \forall x', x'' \in [a,b]:$

$|x'-x''|<\delta' \Rightarrow |f(x')-f(x'')|<\varepsilon$

$\left.\begin{array}{l}|x_{i-1}-\xi_i| \le \Delta x_i \le \delta_\tau < \delta \\ |x_i-\xi_i| \le \Delta x_i \le \delta_\tau<\delta\end{array}\right\} \Rightarrow \left\{\begin{array}{l}|f(x_i)-f(\xi_i)|<\varepsilon \\ |f(x_{i-1})-f(\xi_i)|<\varepsilon\end{array}\right\}(2)$

$\star < \pi\sum_{i=1}^{n} 2\varepsilon\sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i \star\star$

• $\sqrt{1+[f'(x)]^2}$ е непр. в/у $[a,b] \to$ е огр в $[a,b]$, т.е. $\exists\ C>0:$

$\forall x\in[a,b]: \sqrt{1+[f'(x)]^2} \le C\ (3)$

$\overset{(3)}{\Rightarrow} \star\star < 2\pi\varepsilon C\sum_{i=1}^{n}\Delta x_i = \underbrace{2\pi(b-a)\cdot C}_{const}\cdot\varepsilon < [1+2\pi(b-a)C]\,\varepsilon$

$S(P(f)) = \lim S(P(L_\tau)) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

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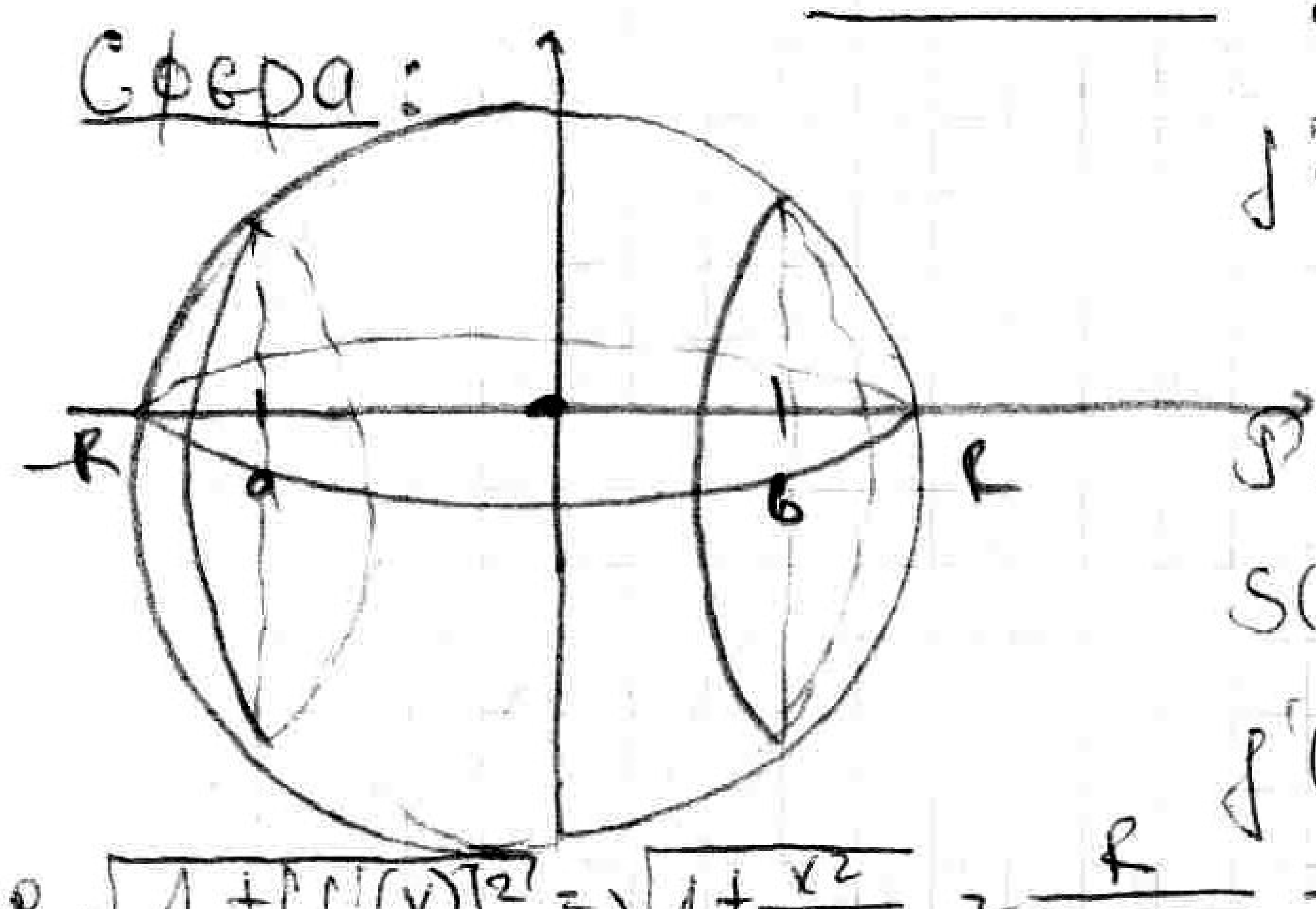
Сфера:



$f: x^2+y^2=R^2 \Rightarrow y=f(x)=\sqrt{R^2-x^2}$,
$x\in[a,b]\subset[-R,R]$

$P(f)$ – сфера с рад. $R$ и ц-р $(0,0)$

$S(P(f)) = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

$f'(x) = \left(\sqrt{R^2-x^2}\right)' = -\frac{2x}{2\sqrt{R^2-x^2}} = -\frac{x}{\sqrt{R^2-x^2}}$

• $\sqrt{1+[f'(x)]^2} = \sqrt{1+\frac{x^2}{R^2-x^2}} = \frac{R}{\sqrt{R^2-x^2}} \Rightarrow$

$f(x)\sqrt{1-[f'(x)]^2} = \cancel{\sqrt{R^2-x^2}}\cdot\frac{R}{\cancel{\sqrt{R^2-x^2}}} = R$

$\Rightarrow S(P(f)) - 2\pi\int_a^b R\,dx = 2\pi R(b-a)$ – лице сферичен пояс (зацур)

• Ако $[a,b]=[-R,R]$ получ. лицето на сфера

$$\boxed{S_{сф} = 4\pi R^2}$$

(11) Несобствени интеграли в/у безкраен интеграл и от неогр. ф-я – опр., с-ва.

Примери: 1)

$f(x) = \frac{1}{1+x^2}$, $x\in[0,+\infty)$

$D = \{(x,y): x\ge 0,\ 0\le y\le \frac{1}{1+x^2}\}$ – неогр. м-во зацур.

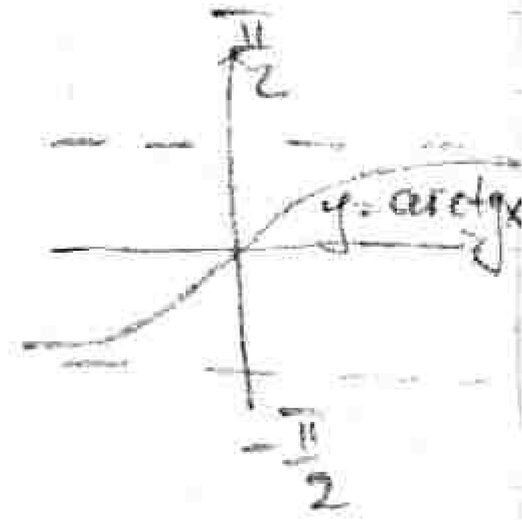
$S(D) = ?$

$0<\xi \to$ произв. число

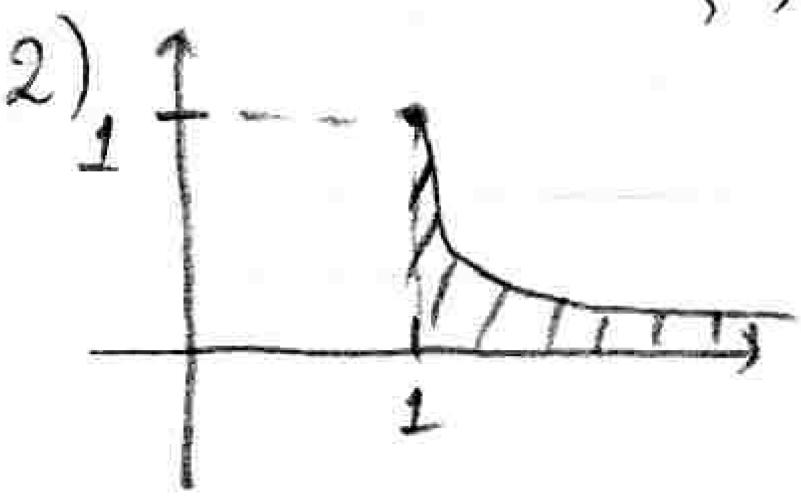
$D_\xi = \{(x,y): 0 \le x \le \xi,\ 0 \le y \le \frac{1}{1+x^2}\}$

$S(D_\xi) = S(\xi)$

$S(\xi) = \int_0^\xi \frac{1}{1+x^2}\,dx = \operatorname{arctg}\Big|_0^\xi = \operatorname{arctg}\xi - \operatorname{arctg}0 = \operatorname{arctg}\xi$




$\lim S(D_\xi) = \lim\limits_{\xi \to +\infty} S(\xi) = \lim\limits_{\xi\to+\infty} \operatorname{arctg}\xi = \boxed{\frac{\pi}{2} = S(D)}$

2) 

$f(x) = \frac{1}{x}$, $[1;+\infty)$

$D = \{(x,y): x \ge 1,\ y \in [0; \frac{1}{x}]\}$

$1 \le \xi$ - произв.

$D_\xi = \{(x,y): 1 \le x \le \xi,\ 0 \le y \le \frac{1}{x}\}$

$\Rightarrow \exists\ S(\xi) = S(D_\xi) = \int_0^\xi \frac{1}{x}\,dx = \lim\limits_{\xi\to+\infty} \frac{1}{x}\Big|_1^\xi = \ln\xi - \ln 1 = \ln\xi$

$\lim\limits_{\xi\to+\infty} S(D_\xi) = \lim\limits_{\xi\to+\infty} S(\xi) = \lim\limits_{\xi\to+\infty} \ln\xi = +\infty$

$\Rightarrow \nexists$ граница

$\Rightarrow$ лицето на защр. част не може да се намери

Оп| Нека $f(x)$ - деф в/у $[a,+\infty)$ и такова, че:

$\forall \xi \ge a$, $f(x)$ е инт. в/у $[a;\xi]$

- Ако $\exists \lim\limits_{\xi\to+\infty} \int_a^\xi f(x)\,dx$, то казваме, че $f(x)$ е инт. в несобствен смисъл в/у $[a,b]$.

$\lim\limits_{\xi\to+\infty} \int_a^\xi f(x)\,dx = \int_a^{+\infty} f(x)\,dx$ - несобствен интеграл (сходящ се) от $f(x)$ в/у $[a;+\infty)$

Примери: 1) $f(x) = \frac{1}{1+x^2}$ е инт. в н. см. в/у $[a,b]$

$\int_a^{+\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$

2) $f(x) = \frac{1}{x}$, $[1;+\infty)$ - $f(x)$ не е инт. в н. см. в/у $[1;+\infty)$

Забележка: $f(x)$ деф в/у $[a;+\infty)$ и инт. в/у $[a;\xi]$ $\forall a \le \xi$,

$\int_a^{+\infty} f(x)\,dx$ $\quad \int_a^{+\infty} \frac{1}{x}\,dx$

Оп| 1) $f(x)$ - деф в/у $[-\infty;a]$ и инт. в/у $[\eta,a]$ $\forall \eta \le a$

- Ако $\exists \lim\limits_{\eta\to-\infty} \int_\eta^a = \int_{-\infty}^a f(x)\,dx$ ↳ нес. инт. в/у $(-\infty;a]$

↳ инт. в н. см. в инт. $[\eta, a]$

2) $f(x)$ - деф в/у $(-\infty,+\infty)$ и инт. $\forall [\eta,\xi]$:

$\forall (\eta \le \xi) \Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_c^{+\infty} f(x)\,dx$, $c \in \mathbb{R}$

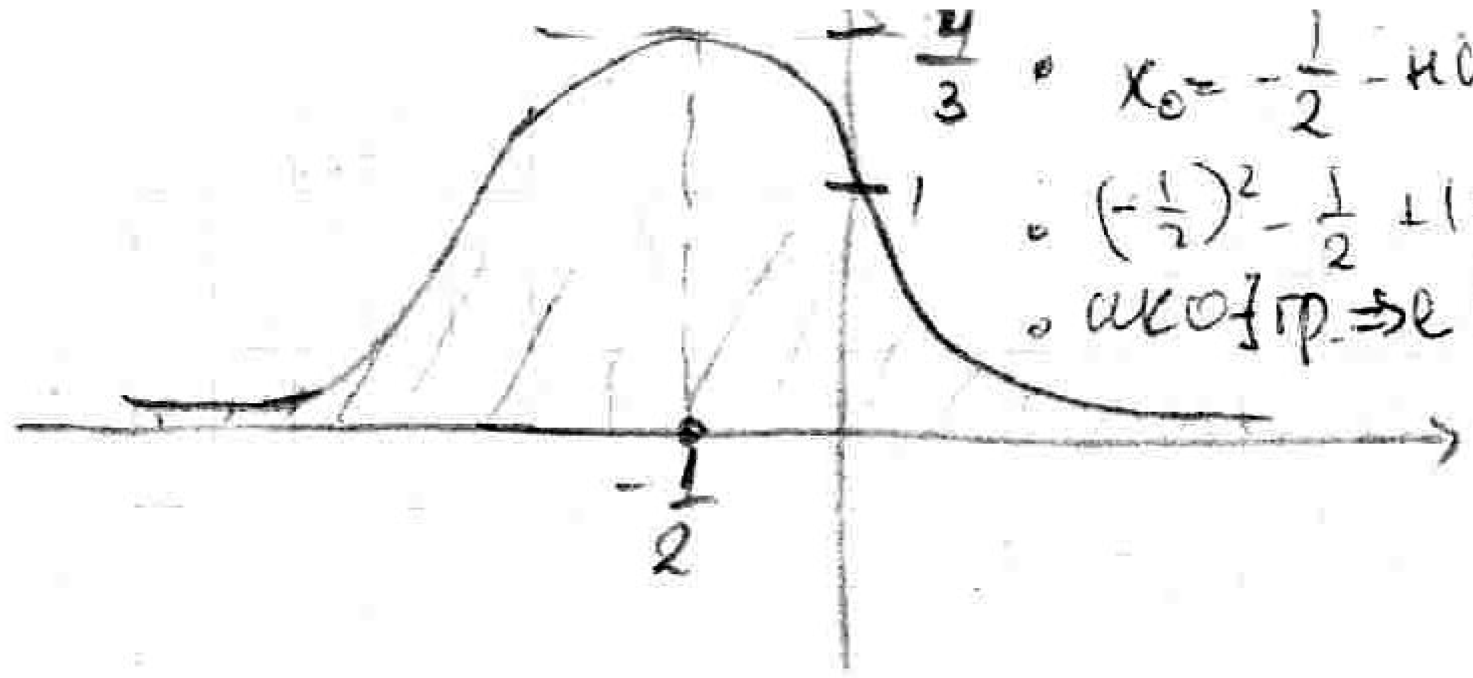
Примери: 1) $\int_{-\infty}^0 x e^{-x^2}\,dx = \lim\limits_{\eta\to-\infty} \int_\eta^0 x e^{-x^2}\,dx =$

$= \lim\limits_{\eta\to-\infty} \left[-\frac{1}{2}\int_\eta^0 e^{-x^2}\,d(-x^2)\right] = \lim\limits_{\eta\to-\infty} \left(-\frac{1}{2}\right) e^{-x^2}\Big|_\eta^0 = \left(\frac{1}{e^{\eta^2}} \to 0\right)$

$= \lim\limits_{\eta\to-\infty} \left[-\frac{1}{2} + \frac{1}{2} e^{-\eta^2}\right] = -\frac{1}{2} + \frac{1}{2}\cdot 0 = -\frac{1}{2} \Rightarrow$ е сходящ със ст-ст $-\frac{1}{2}$

2) $\int_{-\infty}^{+\infty} \frac{1}{x^2+x+1}\,dx =$

$= \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty}} \int_{\eta}^{\xi} \frac{1}{x^2+x+1}\,dx =$




• $x_0=-\frac{1}{2}$ - най-ниц

• $(-\frac{1}{2})^2-\frac{1}{2}+1=\frac{4}{3}$

• ако ∫ гр. ⇒ е сх

$= \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty}} \int_{\eta}^{\xi} \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,d(x+\tfrac{1}{2}) =$

$= \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty}} \frac{4}{3}\int_{\eta}^{\xi} \frac{1}{\left[\frac{2}{\sqrt3}(x+\frac12)\right]^2+1}\,d\left[\frac{2}{\sqrt3}(x+\tfrac12)\right] = \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty}} \frac{2}{3}\operatorname{arctg}\left[\frac{2}{\sqrt3}(x+\tfrac12)\right]\Big|_{\eta}^{\xi} =$

$= \frac{2}{\sqrt3}\lim_{\substack{\xi\to+\infty\\ \eta\to-\infty}}\left[\operatorname{arctg}\frac{2}{\sqrt3}(\xi+\tfrac12)-\operatorname{arctg}\frac{2}{\sqrt3}(\eta+\tfrac12)\right] =$

$= \frac{2}{3}\Big[\lim_{\xi\to+\infty} tg\,\underbrace{\frac{2}{\sqrt3}(\xi+\tfrac12)}_{\to+\infty} - \lim_{\eta\to-\infty}\operatorname{arctg}\underbrace{\frac{2}{\sqrt3}(\eta+\tfrac12)}_{\to-\infty}\Big] = \frac{2}{\sqrt3}\left[\frac{\pi}{2}-(-\frac{\pi}{2})\right] = \frac{2\pi}{\sqrt3}$

=> инт. е сх. със ст-ст $\frac{2\pi}{\sqrt3}$ -> лице на заключ. част. -> м/у ф-ята и $Ox$

3) $I(\alpha) = \int_{1}^{+\infty}\frac{1}{x^\alpha}\,dx = \begin{cases}\frac{1}{\alpha-1}, \text{ ако } \alpha>1, \text{ сходящ}\\ \text{разходящ, ако } 0<\alpha\le 1\end{cases}$

$\alpha=1$: $\int_{1}^{+\infty}\frac{1}{x}\,dx$ -> разходящ

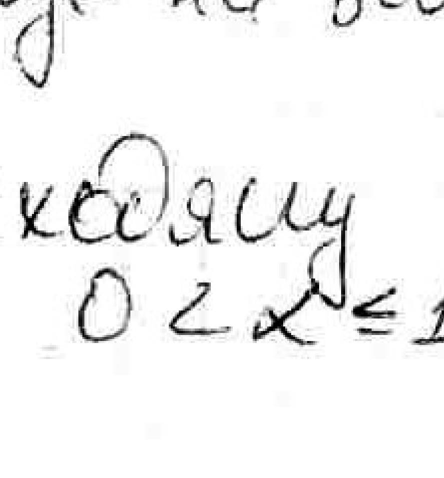
=> $\alpha\ne 1$: $\int_{1}^{+\infty}\frac{1}{x^\alpha}\,dx = \lim_{\xi\to+\infty}\int_{1}^{\xi}\frac{1}{x^\alpha}\,dx = \lim_{\xi\to+\infty}\int_{1}^{\xi}x^{-\alpha}\,dx = \lim_{\xi\to+\infty}\frac{x^{-\alpha+1}}{-\alpha+1}\Big|_{1}^{\xi}$

$= \frac{1}{1-\alpha}\left[\lim_{\xi\to+\infty}\xi^{1-\alpha}-1\right] = \frac{1}{1-\alpha}\left[\lim_{\xi\to+\infty}\frac{1}{\xi^{\alpha-1}}-1\right]$

$\overset{\alpha>1}{=} \frac{1}{1-\alpha}(0-1) = \frac{1}{\alpha-1}$ Ако $\alpha>1$, $I(\alpha)$ е сх

$\alpha\le 1$ $\left[\lim_{\xi\to+\infty}\xi^{1-\alpha}-1\right] = \infty$ => Ако $\alpha\in(0,1)$, то $I(\alpha)$ е разх

4) $f(x)=\frac{1}{\sqrt{1-x}}$, $x\in[0,1)$




$D=\{(x,y):\ 0\le x<1,\ 0\le y\le\frac{1}{\sqrt{1-x}}\}$

$S(D)=?$ лице на безкр. част.

- $0\le\xi<1$
  $D_\xi=\{(x,y):\ 0\le x\le\xi,\ 0\le y\le\frac{1}{\sqrt{1-x}}\}$
- $S(D_\xi)=\int_0^\xi\frac{1}{\sqrt{1-x}}\,dx=\int_0^\xi(1-x)^{-\frac12}\,d(1-x)=-\frac{(1-x)^{\frac12}}{\frac12}\Big|_0^\xi=-2\sqrt{1-\xi}+2$
- $\lim_{\xi\to1-0}S(D_\xi)=\lim_{\xi\to1-0}\int_0^\xi\frac{1}{\sqrt{1-x}}\,dx=\lim_{\xi\to1-0}\left[-2\sqrt{1-\xi}+2\right]=2=S(D)$

=> $\exists\ S(D)=2$

5) $f(x)=\frac{1}{1-x}$, $x\in[0,1)$

$D=\{(x,y):\ 0\le x\le 1,\ 0\le y\le\frac{1}{1-x}\}$

$D_\xi=\{(x,y):\ 0\le x\le\xi,\ 0\le y\le\frac{1}{1-x}\}$

$S(D_\xi)=\int_0^\xi\frac{1}{1-x}\,dx=-\int_0^\xi\frac{1}{1-x}\cdot d(1-x)=-\ln(1-x)\Big|_0^\xi=-\ln(1-\xi)+\ln1=-\ln(1-\xi)$

$\lim\limits_{\xi\to 1-0} S(D_\xi) = \lim\limits_{\xi\to1-0} [-\ln(1-\xi)] = +\infty$ => не можем да нам. лице на D

**Деф** | Нека $f(x)$ е деф. в/у $[a,b)$, $f(x)$ - инт за $\forall: a\le\xi<b$ и неогр. в/у $[\xi, b)$. Тогава, ако $\exists \lim\int_a^\xi f(x)dx$, то казв., че $f(x)$ е инт. в несобств. смисъл в/у $[a,b]$, а $\lim\limits_{\xi\to b-0}\int_a^\xi f(x)dx = \int_a^b f(x)dx$ -> несобств. инт. от $f(x)$ в/у $[a,b]$

- с особ. в долната гр.
- $\int_0^1 \frac{1}{\sqrt{1-x}}dx$ - несобств. инт. , $\frac{1}{1-x}$ - не е инт. в несобств. смисъл

**Забележка** | $\int_a^b f(x)dx$: 1) $f(x)$ - деф. $[a,b]$ и $\forall\ \eta\in[a,b]$: $f(x)$ е инт. $[\eta, b]$ и неогр. в/у $[a,\eta]$ -> $\lim\limits_{\eta\to a+0}\int_\eta^b f(x)dx = \int_a^b f(x)dx$ -> горната гр.

2) Ако $f(x)$ е деф в $(a,b)$ и $\forall\ a<\eta\le\xi<b$: $f(x)$ е инт. $[\eta,\xi]$ и неогр. в/у $[a,\eta]$ и $[\xi, b)$ -> $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ c\in(a,b)$

е особ. в горна и долна граница

- $\int_a^{\infty} f(x)dx$ - несобств. $\qquad \int_a^b f(x)dx$ - тр. да пров. дали е несобств.

Примери:

1) $\int_0^1 \frac{1}{x^\alpha}dx = \begin{cases} \text{разх}, & \alpha\ge1 \\ \text{сх}, & 0<\alpha<1\end{cases} \qquad \int_1^{+\infty}\frac{1}{x^\alpha}dx = \begin{cases}\text{сх}, & \alpha>1 \\ \text{разх}, & 0<\alpha\le 1\end{cases}$

$\lim\limits_{\eta\to0+0}\frac{1}{x^\alpha}dx = \lim\limits_{\eta\to0+0}\int_\eta^1 x^{-\alpha}dx \overset{\alpha\ne1}{=} \lim\limits_{\eta\to0+0}\frac{x^{-\alpha+1}}{-\alpha+1}\Big|_\eta^1 = \frac{1}{1-\alpha}$

$= \frac{1}{1-\alpha}\left[\lim\limits_{\eta\to0+0}\{1-\eta^{-\alpha+1}\}\right] \begin{cases}\xrightarrow{\alpha>1}\infty \\ \xrightarrow{\alpha<1} \frac{1}{1-\alpha}\end{cases}$

=> $\left| \begin{array}{l} \alpha>1 \quad \int_0^1\frac{1}{x^\alpha}dx \text{ е разх} \\ 0<\alpha<1, \int_0^1\frac{1}{x^\alpha}dx \text{ е сх. и } \int_0^1\frac{1}{x^\alpha}dx = \frac{1}{1-\alpha}\end{array}\right.$

- $\alpha=1$, $\lim\limits_{\eta\to0+0}\int_0^1\frac{1}{x}dx = \lim\limits_{\eta\to0+0}\ln x\Big|_\eta^1 = \lim\limits_{\eta\to0+0}(\ln1-\ln\eta) = \infty$

=> $\int_0^1\frac{1}{x^\alpha}dx$ е разх.

- $\int_a^b\frac{1}{(b-x)^\alpha} = \begin{cases}\text{разх} & \alpha\ge1 \\ \text{сх}, & 0<\alpha<1, \left(\frac{(b-\alpha)^{1-\alpha}}{1-\alpha}\right)\end{cases}$

## Свойства на несобств. интеграл

**Свойство 1** | Ако н. инт. $\int_a^{+\infty}f(x)dx$ и $\int_a^{+\infty}g(x)dx$ са сх => $\int_a^{+\infty}[f(x)+g(x)]dx$ и $\int_a^{+\infty}\lambda f(x)dx\ (\lambda\in\mathbb{R})$ са сх. и: 1) $\int_a^{+\infty}[f(x)+g(x)]dx = \int_a^{+\infty}f(x)dx + \int_a^{+\infty}g(x)dx$.

2) $\int_a^{+\infty}\lambda f(x)dx = \lambda\int_a^{+\infty}f(x)dx,\ \lambda\in\mathbb{R}$

Д-во:

1) $\lim\limits_{\xi\to+\infty}\int_a^\xi [f(x)+g(x)]dx = \lim\limits_{\xi\to+\infty}[\int_a^\xi f(x)dx + \int_a^\xi g(x)dx] =$

$\int_a^{+\infty}[f(x)+g(x)]dx = \lim\limits_{\xi\to+\infty}\int_a^\xi f(x)dx + \lim\limits_{\xi\to+\infty}\int_a^\xi g(x)dx = \int_a^{+\infty} f(x)dx + \int_a^{+\infty} g(x)dx$

инт. са сх => ∃ lim

2) $\lim\limits_{\xi\to+\infty}\int_a^\xi \lambda f(x)dx = \lim\limits_{\xi\to+\infty}\lambda\int_a^\xi f(x)dx = \lambda\lim\limits_{\xi\to+\infty}\int_a^\xi f(x)dx = \lambda\int_a^{+\infty} f(x)dx$

$\int_a^{+\infty}\lambda f(x)dx$

<u>Свойство 2</u>| Нека $f(x)$ е непр. в/у $[a,+\infty)$ и $F(x)$ е прим. ф-я на $f(x)$ в/у $[a,+\infty)$. Несобств. инт. $\int_a^{+\infty} f(x)dx$ е сх. <=> $\exists \lim\limits_{x\to+\infty}F(x) = F(+\infty)$

$\lim\limits_{\xi\to+\infty}\int_a^\xi f(x)dx = \lim\limits_{\xi\to+\infty}F(x)\Big|_a^\xi \overset{\text{д-во:}}{=} \lim\limits_{\xi\to+\infty}[F(\xi)-F(a)] = \lim\limits_{\xi\to+\infty}F(x)-F(a) =$

$= F(+\infty)-F(a) = F(x)\Big|_a^{+\infty}$

<u>Пример</u>: $\int_a^{+\infty}\frac{arctg x}{1+x^2}dx = \int_0^{+\infty} arctg x\, d\, arctg x = \frac{arctg^2 x}{2}\Big|_0^{+\infty} =$

$= \lim\limits_{x\to+\infty}\frac{arctg^2 x}{2} - \frac{arctg^2 0}{2} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - 0 = \frac{\pi^2}{8}$

<u>Свойство 3</u>| Нека $f(x)$ и $g(x)$ са ф-и в/у $[a,+\infty)$ и имат непр. произв. $f'(x)$ и $g'(x)$ в/у $[a,+\infty)$. Ако $\exists \lim\limits_{x\to+\infty} f(x)g(x)$ и $\int_a^{+\infty} g(x)df(x)$ е сх. => е сх. $\int_a^{+\infty} f(x)g(x)$ и $\int_a^{+\infty} f(x)dg(x) =$

$= f(x)g(x)\Big|_a^{+\infty} - \int_a^{+\infty} g(x)df(x)$ (Т-ма на ЛОК за инт. по части)

$\lim\limits_{\xi\to+\infty}\int_a^\xi f(x)dg(x) \overset{\text{д-во:}}{=} \lim\limits_{\xi\to+\infty}[f(x)g(x)\Big|_a^\xi - \int_a^\xi g(x)f(x)] =$

$= \lim\limits_{\xi\to+\infty}[f(x)g(x) - f(a)g(a)] - \lim\limits_{\xi\to+\infty}\int_a^\xi g(x)df(x) =$

$= (f(+\infty).g(+\infty) - f(a).g(a)) - \int_a^{+\infty} g(x)df(x) =$

$= f(x)g(x)\Big|_a^{+\infty} - \int_a^{+\infty} g(x).df(x)$

<u>Пример</u>: $\int_0^{+\infty} x e^{-x}dx =$

$= -\int_0^{+\infty} x e^{-x}d(-x) = -\int_0^{+\infty} d e^{-x} = -[x e^{-x}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-x}dx]$

$= -[x.e^{-x}\Big|_0^{+\infty} + e^{-x}\Big|_0^{+\infty}] =$

$= -[\lim\limits_{x\to+\infty} x.e^{-x} - 0.e^{-0} + \lim\limits_{x\to+\infty} e^{-x} - e^{-0}] =$

$= -[0 - 0 + 0 - 1] = 1$

<u>Свойство 4</u>| Нека ф-та $f(x)$ е непр. в/у $[a,+\infty)$ и ф-та $x=\varphi(t)$:

1) $\varphi: [\alpha,\beta) \to [a,+\infty)$
2) $\varphi(\alpha) = a$, $\lim\limits_{t\to\beta-0}\varphi(t) = +\infty$
3) $\exists\ \varphi'(t)$ - непр. в/у $[\alpha,\beta)$

$\Rightarrow \int_a^{+\infty} f(x)dx = \int_\alpha^\beta f(\varphi(t)).\varphi'(t)dt = \int_\alpha^\beta f(\varphi(t))d\varphi(t)$

Ако $\int_a^{+\infty} f(x)\,dx$ е сх!

Д-во:

$\lim\limits_{\xi\to+\infty}\int_a^{\xi} f(x)dx = \lim\limits_{\xi\to+\infty}\int_{\alpha}^{\varphi(t_\xi)} f(\varphi(t))\varphi'(t)\,dt = *$

$\exists\ t_\xi \in [\alpha,\beta) : \varphi(t_\xi)=\xi$

$* = \lim\limits_{t'\to\beta}\int_{\alpha}^{t'} f(\varphi(t)).\varphi'(t)\,dt = \int_{\alpha}^{\beta} f(\varphi(t)).\varphi'(t)\,dt$

Пример: $\int_0^{+\infty}\frac{1}{(x^2+1)^{\frac{3}{2}}}dx = \int_0^{\pi/2}\frac{1}{\frac{1}{\cos^3 t}}dt = \int_0^{\pi/2}\cos t\,dt = *$ собств. инт.

$x=tg\,t \quad 0\le x<+\infty, \quad 0\le t<\frac{\pi}{2}$

$x^2+1 = tg^2 t+1 = \frac{1}{\cos^2 t}$

$dx = d\,tg\,t = \frac{1}{\cos^2 t}dt$

$* = \sin t\,\Big|_0^{\pi/2} = 1-0 = 1$

Свойство 5| Нека $f(x)\le g(x)$, $\forall x\in[a,+\infty)$ и инт. $\int_a^{+\infty} f(x)dx$, $\int_a^{+\infty} g(x)dx$ са сх $\Rightarrow \int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx$

Д-во:

Т.к. $f(x)\le g(x) \Rightarrow \forall a\le\xi : \int_a^{\xi} f(x)dx \le \int_a^{\xi} g(x)dx \Rightarrow$

$\lim\limits_{\xi\to+\infty}\int_a^{\xi} f(x)dx \le \lim\limits_{\xi\to+\infty}\int_a^{\xi} g(x)dx \Rightarrow \int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx$

## 12. Несобствен интеграл от неотрицателна функция.

## Абсолютно и условно сходящи се несобствени интеграли.

Т| Нека $f(x)\ge 0$ и инт. в/у $\forall$ под.инт. $[a,\xi]$, $\forall \xi\ge a$ в/у $[a,+\infty)$

Несобств. инт. $\int_a^{+\infty} f(x)dx$ е сх $\Leftrightarrow F(\xi)=\int_a^{\xi} f(x)dx$ е огр. в/у $[a,+\infty)$

При това $\int_a^{+\infty} f(x)dx = \sup\limits_{\xi\in[a,+\infty)} F(\xi)$

Д-во:

Лема: Нека $g(x)$ е мон. раст. в/у $[a,+\infty)$

Тогава $\exists \lim\limits_{x\to+\infty} g(x) \Leftrightarrow g(x)$ е огр. в/у $[a,+\infty)$

При това $\lim\limits_{x\to+\infty} g(x) = \sup\limits_{x\in[a,+\infty)} g(x)$

$F(\xi)=\int_a^{\xi} f(x)dx \Rightarrow$ мон. раст. в/у $[a,+\infty)$

Действ. нека $a\le\xi_1\le\xi_2$

Разгл. $F(\xi_2)-F(\xi_1) = \int_a^{\xi_2} f(x)\,dx - \int_a^{\xi_1} f(x)dx = \int_{\xi_1}^{\xi_2} f(x)\,dx \underset{\text{с-ва опр. инт.}}{\ge} 0$

$\Rightarrow F(\xi_2)\ge F(\xi_1)$, т.е. $F(x)$ е м.↑ в/у $[a,+\infty)$

$\int_a^{+\infty} f(x)\,dx$ е сх $\overset{def}{\Leftrightarrow} \exists \lim\limits_{\xi\to+\infty} F(\xi) \Leftrightarrow F(\xi)$ е огр. в/у $[a,+\infty)$

При това $\int_a^{+\infty} f(x)\,dx = \lim\limits_{\xi\to+\infty} F(\xi) \overset{\text{Л}}{=} \sup\limits_{\xi\in[a,+\infty)} F(\xi) = \sup\limits_{\xi\in[a,+\infty)} \int_a^{\xi} f(x)\,dx$

**Т1** (Т-ма за ср.) Нека $f(x)$ и $g(x)$ са деф. в/у $[a,+\infty)$ и инт. в/у $\forall\,[a,\xi]\ (\forall \xi \geq a)$ и за $\forall x \in [a,+\infty)$, $0 \leq f(x) \leq g(x)$

Тогава:

1) ако $\int^{+\infty} g(x)dx$ е сх. $\Rightarrow \int^{+\infty} f(x)dx$ - сх.

($\overset{(1)}{\Rightarrow}$ и доп. на прот.) 2) ако $\int_a^{+\infty} f(x)dx$ е разх. $\Rightarrow \int_a^{+\infty} g(x)dx$ е разх

<u>Д-во:</u>

Нека $\int_a^{+\infty} g(x)dx$ е сх.

$\Rightarrow G(\xi) = \int_a^{\xi} g(x)dx$ е огр. в/у $[a,+\infty)$

$\Rightarrow F(\xi) = \int_a^{\xi} f(x)dx$ имаме, че $F(\xi) \leq G(\xi)$, $(\forall \xi \geq a)$

т.к. $F(\xi) = \int_a^{\xi} f(x)dx \leq \int_a^{\xi} g(x)dx = G(\xi)$

• $G(\xi)$ - огр. $\Rightarrow \exists\, M > 0 : \forall \xi \geq a \Rightarrow G(\xi) \leq M \Rightarrow \forall \xi \geq a$:

$F(\xi) \leq G(\xi) \leq M \Rightarrow F(\xi)$ е огр. в/у $(a,+\infty)$

$\Rightarrow \int_a^{+\infty} f(x)dx$ е сх

<u>Примери:</u> (изсл. за сх.)

1) $J(\alpha) = \int_1^{+\infty} \frac{\sin^2 x}{x^\alpha}dx\ (\alpha > 0)$

$0 \leq f(x) = \frac{\sin^2 x}{x^\alpha}$, $x \in [1,+\infty)$

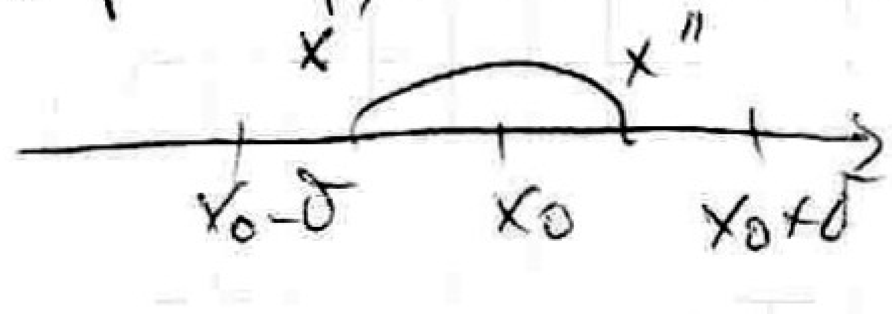
$0 < \frac{\sin^2 x}{x^\alpha} \leq \frac{1}{x^\alpha}$, $\int_1^{+\infty} \frac{1}{x^\alpha}dx$ е сх., ако $\alpha > 0$

съгл. т-ма за ср. $\Rightarrow \int_0^{\infty} \frac{\sin^2 x}{x^2}dx$ е сх, ако $\alpha > 1$

**Лема:** $\exists \lim_{x\to+\infty} f(x)$ (деф в/у $(a,+\infty)$) $\Leftrightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$:

$\forall x', x'' \in (\delta,+\infty) \Rightarrow |f(x') - f(x'')| < \varepsilon$

(Критерий Коши: $\exists \lim_{x\to x_0} f(x) \Leftrightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall x', x'' \in$



$x' \neq x_0$, $x'' \neq x_0$, $|x_0 - x'| < \delta$, $|x_0 - x''| < \delta \Rightarrow$

$|f(x') - f(x'')| < \varepsilon$ )

**Т6** $\nexists \lim_{x\to+\infty} f(x) \Leftrightarrow \exists\ \varepsilon_0 > 0 : \forall \delta > 0, \exists\ x_\delta', x_\delta'' \in (\delta,+\infty)$

$|f(x_\delta') - f(x_\delta'')| \cdot < \varepsilon_0$

**Т6** $\int_a^{+\infty} f(x)dx$ - разх $\Leftrightarrow \nexists \lim_{\xi\to+\infty} F(\xi)$, $F(\xi) = \int_a^{\xi} f(x)dx \Leftrightarrow$

$\exists\, \varepsilon_0 > 0, \forall \delta > 0, \exists\, \xi', \xi'', \xi' > \delta, \xi'' > \delta, (\xi', \xi'' \in (\delta,+\infty))$: $|F(\xi') - F(\xi'')| \geq \varepsilon_0$

**Пр** • $\int_0^{+\infty} \frac{\sin^2 x}{x^\alpha}dx$, $\alpha \leq 1$, $\forall \delta > 0, \exists\ \xi_\delta' = n\pi$, $\xi_\delta'' = 2n\pi$ $(n\pi > \delta)$ $(n > \frac{\delta}{\pi})$

• $x^\alpha < x$, $n\pi \leq x \leq 2n\pi \Rightarrow \frac{1}{x} > \frac{1}{2n\pi}$

$\Rightarrow \left|\int_{\xi'}^{\xi''} \frac{\sin^2 x}{x^\alpha}dx\right| = \left|\int_{n\pi}^{2n\pi} \frac{\sin^2 x}{x^\alpha}dx\right| = \int_{n\pi}^{2n\pi} \frac{\sin^2 x}{x^\alpha}dx \geq \int_{n\pi}^{2n\pi} \frac{\sin^2 x}{x}dx \geq$

$\geq \int_{n\pi}^{2n\pi} \frac{\sin^2 x}{2n\pi}dx = \frac{1}{2n\pi}\int_{n\pi}^{2n\pi} \sin^2 x\,dx = \frac{1}{2n\pi}\int_{n\pi}^{2n\pi} \frac{1-\cos 2x}{2}dx = \frac{1}{4n\pi}\left[\int_{n\pi}^{2n\pi} 1dx - \frac{1}{2}\int_{n\pi}^{2n\pi} \cos 2x\, d(2x)\right.$

$= \frac{1}{4n\pi}\Big[\underbrace{n\pi - \frac{1}{2}\sin 2x\Big|_{n\pi}^{2n\pi}}_{0}\Big] = \frac{1}{4} = \varepsilon_0$

$\Rightarrow \int_0^{+\infty} \frac{\sin^2 x}{x^\alpha}dx$ – разх, $0<\alpha\le 1$

Деф| Несобствен интеграл $\int_a^{+\infty} f(x)dx$ се нарича:

1) абсолютно сходящ, ако е сх $\int_a^{+\infty}|f(x)|dx$

2) условно сходящ, ако $\int_a^{+\infty} f(x)$ е сх. и $\int_a^{+\infty}|f(x)|dx$ е разх.

Т| Ако несоб. инт. $\int_a^{+\infty} f(x)dx$ е абс. сх. $\Rightarrow \int_a^{+\infty} f(x)dx$ е сх.

Д-во:

т.к. $\int_a^{+\infty}f(x)dx$ е абс. сх. $\xRightarrow{D} \int_a^{+\infty}|f(x)|dx$ е сх. $\xRightarrow{кр.К.}$

$\forall \varepsilon>0, \exists \delta>0: \forall \xi', \xi''>\delta \Rightarrow \left|\int_{\xi'}^{\xi''}|f(x)|dx\right|<\varepsilon \xRightarrow{св. на опр. инт.}$

$\left|\int_{\xi'}^{\xi''} f(x)dx\right| \le \left|\int_{\xi'}^{\xi''}|f(x)|dx\right| < \varepsilon \xRightarrow{кр.К.}$

$\int_a^{+\infty} f(x)dx$ е сх. т.к. $\exists \lim_{\xi\to+\infty}\int_a^{\xi} f(x)dx$

Пример| $\int_1^{+\infty}\frac{\sin x}{x^\alpha}dx\ (0<\alpha)$

1) $\alpha>1$ $\quad 0\le\left|\frac{\sin x}{x^\alpha}\right| = \frac{|\sin x|}{x^\alpha}\le\frac{1}{x^\alpha}$, $x\in(1,+\infty)$,

т.к. $\int_1^{+\infty}\frac{1}{x^\alpha}dx$ е сх. при $\alpha>1 \xRightarrow{п-п за сравн.}$

$\int_1^{+\infty}\frac{|\sin x|}{x^\alpha}dx$ е сх $\Rightarrow \int_1^{+\infty}\frac{\sin x}{x^\alpha}dx$ е абс. сх.

2) $0<\alpha\le 1$ $\quad \int_1^{+\infty}\frac{|\sin x|}{x^\alpha}dx$ – разх.

$\int_1^{+\infty}\frac{\sin x}{x^\alpha}dx = -\int_1^{+\infty}\frac{1}{x^\alpha}d\cos x = -\frac{\cos x}{x^\alpha}\Big|_1^{+\infty} + \int_1^{+\infty}\cos x\, d\frac{1}{x^\alpha}$

$= \cos 1 + (-\alpha)\int_1^{+\infty}\frac{\cos x}{x^{\alpha+1}}dx$

$\lim_{x\to+\infty}\frac{\cos x}{x^\alpha}=0$

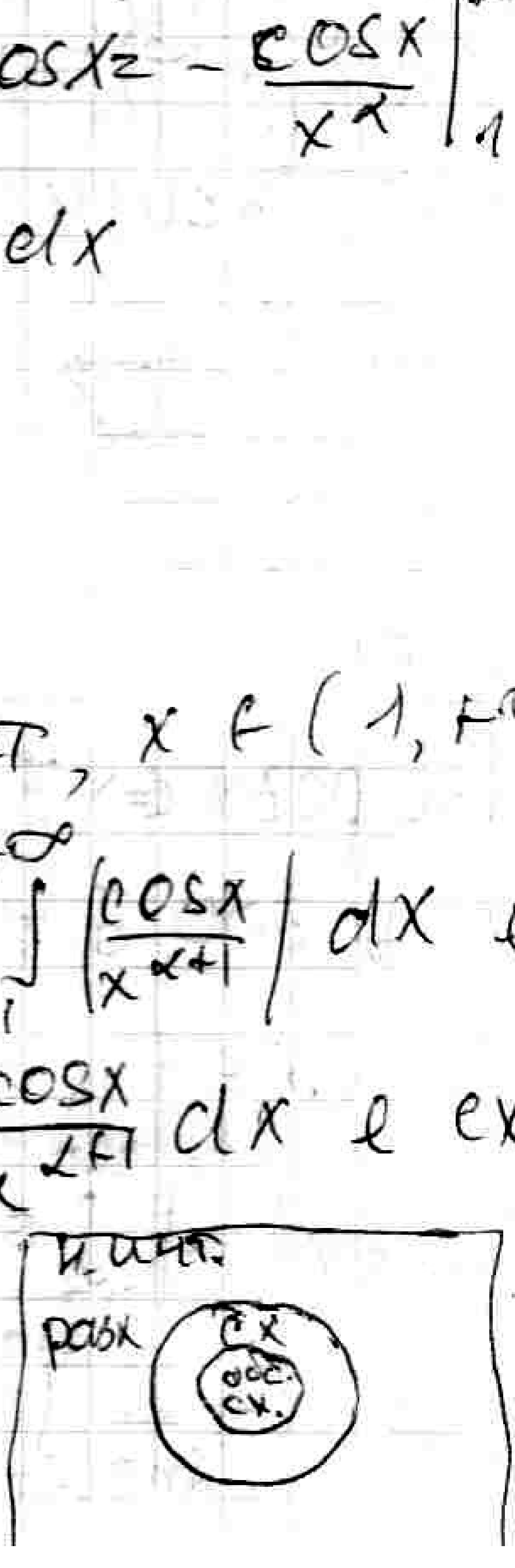
$\int_1^{+\infty}\frac{\cos x}{x^{\alpha+1}}dx$ е сх

Разгл. $\int_1^{+\infty}\left|\frac{\cos x}{x^{\alpha+1}}\right|dx \Rightarrow 0\le \left|\frac{\cos x}{x^{\alpha+1}}\right|\le\frac{1}{x^{\alpha+1}}$, $x\in(1,+\infty)$

$\int_1^{+\infty}\frac{1}{x^{\alpha+1}}dx$ е сх., $\alpha+1>1 \Rightarrow \int_1^{+\infty}\left|\frac{\cos x}{x^{\alpha+1}}\right|dx$ е сх.

$\Rightarrow \int_1^{+\infty}\frac{\cos x}{x^{\alpha+1}}dx$ е абс. сх. $\Rightarrow \int_1^{+\infty}\frac{\cos x}{x^{\alpha+1}}dx$ е сх. $\Rightarrow$

$\int_1^{+\infty}\frac{\sin x}{x^\alpha}dx$ е сходящ



**Тт** Ако $\int_a^{+\infty} f(x)\,dx$ и $\int_a^{+\infty} g(x)\,dx$ са абс. сх. $\Rightarrow$

1) $\int_a^{+\infty} [f(x)+g(x)]\,dx$ е абс. сх.

2) $\int_a^{+\infty} \lambda f(x)\,dx$ е абс. сх. ($\forall \lambda \in \mathbb{R}$)

**Д-во:**

1) $\int_a^{+\infty} f(x)\,dx + \int_a^{+\infty} g(x)\,dx$ са абс. сх. $\Rightarrow$ са сх.

$\int_a^{+\infty} |f(x)|\,dx$ и $\int_a^{+\infty} |g(x)|\,dx \Rightarrow \int_a (|f(x)|+|g(x)|)\,dx$

т.к. $0 \le |f(x)+g(x)| \le |f(x)| + |g(x)|$ $\forall x \in [a,+\infty)$ и

т.к. $\int_a^{+\infty} (|f(x)|+|g(x)|)\,dx$ е сх $\xrightarrow[\text{сравн.}]{\text{п-п. за}}$ $\int_a^{+\infty} |f(x)+g(x)|\,dx$ е сх

$\Rightarrow \int_a^{+\infty} (f(x)+g(x))\,dx$ е сх. абс.

2) $\int_a^{+\infty} f(x)\,dx$ е абс. сх. $\Rightarrow \int_a^{+\infty} |f(x)|\,dx$ е сх $\Rightarrow$

$\int_a^{+\infty} |\lambda| |f(x)|\,dx = \int_a^{+\infty} |\lambda f(x)|\,dx$ е сх $\Rightarrow \int_a \lambda f(x)\,dx$ е абс. сх.

## 13. Безкрайни числови редове - сходимост, свойства.

1) $1, 2, 3, 4, \dots$ $\quad 1+2+3+4+5$ $\quad S_1 = 1$

2) $1, -1, 1, -1, \dots$ $\quad 1+(-1)+1+(-1)+\dots = 0$ $\quad S_2 = 0$ - разх. $\Rightarrow \sum_{n=1} (-1)^{n-1}$ е разх

3) $1, 2, 3, 0, 0, 0, \dots$ $\quad 1+2+3+0+0+\dots = 6$ $\quad S_3 = 6$

4) $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$ $\quad S_n = 1 + \frac{1}{2} + \dots + \frac{1}{2^n} + \dots = 2\left(1 - \frac{1}{2^n}\right) = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2 \to$ сума на д.ч.р. $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$

**Деф** Нека $a_1, a_2, \dots, a_n, \dots$ е д.ч.р

формална сума: $a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$ $\quad$ $a_n$ $\underline{\text{ел. нар. общ / неин член на д.ч. ред.}}$

- безкраен числов ред

Нека имаме д.ч. ред $\sum_{n=1} a_n$

$\forall n \in \mathbb{N}: S_n = a_1 + a_2 + \dots + a_n \to$ $n$-та парциална сума на д.ч.ред $\sum_{n=1} a_n$

$S_1, S_2, \dots, S_n, \dots \to S$

$S = \sum_{n=1} a_n$ - сх - сума на д.ч. ред

* $S_n = 1+2+\dots+n = \frac{(n+1)n}{2}$

$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{n(n+1)}{2} = +\infty \Rightarrow \sum_{n=1}^{n} n$ е разх.

**Пример:** 1) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} + \dots$

$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$

$\frac{1}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$\lim_{n\to+\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

2) $\sum_{n=0}^{\infty} q^n, (q \in \mathbb{R})$

$(q \neq \pm 1)$ $S_n = 1 + q + q^2 + \dots + q^{n-1} = \frac{1-q^n}{1-q}$

$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1}{1-q}\left(1 - \lim_{n\to\infty} q^n\right) = \begin{cases} \frac{1}{1-q}, & |q|<1 \\ \text{разх}, & |q| \geq 1 \end{cases}$

$\lim_{n\to\infty} q^n = \begin{cases} 0, & |q|<1 \\ \nexists\infty, & |q|>1 \end{cases}$

$q = 1 \Rightarrow \sum_{n=0}^{\infty} (-1)^{n-1} = 1 + (-1) + 1 + (-1) + \dots \rightarrow$ разх.

$q = -1 \Rightarrow \sum_{n=0}^{\infty} 1^n = \sum_{n=1}^{\infty} 1$ – разх

$S_n = \underline{1 + 1 + \dots + 1} = n \rightarrow +\infty$

$\sum_{n=0}^{\infty} q^n = \begin{cases} \frac{1}{1-q}, & \text{ако } |q|<1 \\ \text{разх}, & \text{ако } |q| \geq 1 \end{cases}$

__НУ-ЗП__ | Ако д.т. ред $\sum_{n=1}^{\infty} a_n$ е сх $\Rightarrow \lim_{n\to\infty} a_n = 0$ ($\nLeftarrow$)

Нека $\sum_{n=1}^{\infty} a_n$ е сх. $\Rightarrow \{S_n\}_{n=1}^{\infty}$ е сх., където $S_n = \sum_{k=1}^{n} a_k$ ($\forall n \in \mathbb{N}$)

$\Rightarrow \exists\, S = \lim_{n\to\infty} S_n$ $\qquad a_n = S_n - S_{n-1}$

$S_n = \underbrace{a_1 + a_2 + \dots + a_{n-1}}_{S_{n-1}} + a_n$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_n = S - S = 0$

__Пример__ | $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, $\forall k < n \Rightarrow \sqrt{k} < \sqrt{n} \Rightarrow \frac{1}{\sqrt{k}} > \frac{1}{\sqrt{n}}$

$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} = n\cdot\frac{1}{\sqrt{n}} = \sqrt{n}$

$\forall n \in \mathbb{N}$ $\underset{\downarrow +\infty}{S_n} > \underset{\downarrow +\infty}{\sqrt{n}}$ $n\to\infty$ $\Rightarrow \{S_n\}^{\infty}$ е разх $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ е разх.

$\Rightarrow a_n = \frac{1}{\sqrt{n}} \xrightarrow[n\to+\infty]{} 0$

- $\sum_{n=1}^{\infty} n$, $a_n = n \not\to 0 \Rightarrow$ разх.

- $\sum_{n=1}^{\infty} (-1)^{n-1}$, $a_n = (-1)^{n-1} \not\to 0 \Rightarrow$ разх.

__Свойство 1__ | Ако д.т. редове $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ са сх $\Rightarrow$

1) $\sum_{n=1}^{\infty} \lambda a_n$ $(\lambda \in \mathbb{R})$ е сх. и $\sum_{n=1}^{\infty} \lambda a_n = \lambda \sum_{n=1}^{\infty} a_n$

2) $\sum_{n=1}^{\infty} (a_n + b_n)$ е сх. и $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

__Д-во:__

1) Нека $\sum_{n=1}^{\infty} a_n = S$, и $S_n = \sum_{k=1}^{n} a_k$

обозначаваме: $S_n' = \sum_{n=1} \lambda a_k$ — n-та парц. сума на б.т. ред $\sum_{n=1} \lambda a_n$

$S_n' = \sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k = \lambda S_n$

$\lim_{n\to\infty} S_n' = \lim_{n\to\infty} \lambda S_n = \lambda \lim_{n\to\infty} S_n = \lambda S \Rightarrow$

i) $\sum_{n=1}^{\infty} \lambda a_n$ е сх.

ii) $\sum_{n=1}^{\infty} \lambda a_n = \lambda S = \lambda \sum_{n=1}^{\infty} a_n$

2) Нека $\sum_{n=1}^{\infty} a_n = S$, $\sum_{n=1}^{\infty} b_n = S_1$, $\hat{S_n} = \sum_{k=1}^{n} a_k$, $S_n' = \sum_{n=1}^{n} b_k$

Разгл. $\sum_{n=1}^{\infty} (a_n + b_n)$ и нека $S_n = \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = S_n + \hat{S_n}$

$\lim_{n\to\infty} \hat{S_n} = \lim_{n\to\infty} (S_n + S_n') = \lim_{n\to\infty} S_n + \lim_{n\to\infty} S_n' = S + S_1 \Rightarrow$

i) $\sum_{n=1}^{\infty} (a_n + b_n)$ е сх.

ii) $\sum_{n=1}^{\infty} (a_n + b_n) = S + S_1 = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

Свойство 2| Б.т. ред $\sum_{n=1}^{\infty} a_n$ е сходящ $\Leftrightarrow \sum_{n=N+1} a_n$, к. $N \in \mathbb{N}$

Д-во

Нека $S_n = \sum_{k=1}^{n} a_k$

$\hat{S_m} = \sum_{k=N+1}^{m} a_k \quad (k_m \geq N+1)$

$\hat{S_m} = \sum_{k=N+1}^{m} a_k = \sum_{k=1}^{m} a_k - \sum_{k=1}^{N} a_k = S_m - S_N$

$\hat{S_m} = S_m - S_N \Rightarrow$ сх. не се нарушава

Свойство 3| Ако редът $\sum_{n=1}^{\infty} a_n$ е сх и $S = \sum_{n=1}^{\infty} a_n \Rightarrow$ е сх. и редът $\sum_{m=1}^{\infty} b_m$, където $b_m = \sum_{k=n_m+1}^{n_{m+1}} a_k$, $n_0 = 0 < n_1 < n_2 < \dots < n_m < \dots$ $(n_m \in \mathbb{N})$

и $\sum_{m=1}^{\infty} b_m = S$

Пример| $\sum_{n=1}^{\infty} (-1)^{n-1} = \underbrace{1+(-1)}_{b_1} + \underbrace{1+(-1)}_{b_2} + \dots$

$\sum_{m=1}^{\infty} b_m = \sum_{n=1}^{\infty} ((-1)^{n-1} + (-1)^n) = \sum_{n=1}^{\infty} 0$ — сх. $\Rightarrow$ обр. тв. не е вярно!

Д-во:

Нека $S_n = \sum_{k=1}^{n} a_k$ и $S = \lim_{n\to\infty} S_n$, т.е. $\sum_{n=1}^{\infty} a_n = S$

Нека $\hat{S_m} = \sum_{k=1}^{m} b_k = \sum_{k=1}^{m} \sum_{k=n_{k-1}+1}^{n_k} a_{n_k} = \sum_{l=1}^{n_m} a_l = S_{n_m}$

$S_{n_1}, S_{n_2}, \dots, S_{n_m}, \dots$ е подред

$S_1, S_2, \dots, S_n, \dots \to S$ $\Big\}$ $S_{n_m} \to S$, т.е.

$\hat{S_m} \xrightarrow[m\to\infty]{} S \Rightarrow \sum_{m=1}^{\infty} b_m$ е сх. и $\sum_{m=1}^{\infty} b_m = S$

Т (Критерий на Коши)| Б.т. ред $\sum_{n=1}^{\infty} a_n$ е сх $\Leftrightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon)$

$\forall n > N$, $p \in \mathbb{N} \Rightarrow |a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon$

Д-во: кр. на к. (с Стариуя)

$\sum_{n=1}^{\infty} a_n$ е сх $\Leftrightarrow$ сх $\{S_n\}_{n=1}^{\infty} \Leftrightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon): \forall n > N, p \in N \Rightarrow$

$|S_{n+p} - S_n| < \varepsilon$

$|S_{n+p} - S_n| = |\sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k| = |\sum_{k=n+1}^{n+p} a_k| = |a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon$

Пример | $\sum_{n=1}^{\infty} \frac{1}{n}$, $a_n = \frac{1}{n} \xrightarrow[n \to \infty]{} 0$ — разх.

Кр. на Коши (отрицанието му) | $\sum_{n=1}^{\infty} a_n$ е разх $\Leftrightarrow \exists \varepsilon_0 > 0, \forall N$, $\exists n_0 > N, p_0 \in N: |a_{n_0+1} + a_{n_0+2} + \dots + a_{n_0+p_0}| \geq \varepsilon_0$

• $\sum_{n=1}^{\infty} \frac{1}{n}$: $\forall N \in N$, $n_0 = N$, $p_0 = N$,

$$\underbrace{\frac{1}{N+1}}_{a_{n_0+1}} + \underbrace{\frac{1}{N+2}}_{a_{n_0+2}} + \underbrace{\frac{1}{N+3}}_{a_{N+3}} + \dots + \underbrace{\frac{1}{N+N}}_{a_{N+N}} > \underbrace{\frac{1}{2N} + \dots + \frac{1}{2N}}_{N} = N \cdot \frac{1}{2N} = \underset{\varepsilon_0}{\frac{1}{2}}$$

$1 \leq k \leq N$

$\frac{1}{N+k} > \frac{1}{2N} \xRightarrow{\text{кр. Коши}} \sum_{n=1}^{\infty} \frac{1}{n}$ е разх.

хармоничен ред!

14. Редове с неотрицателни членове, признак за сравнение. Крит. на Даламбер. Крит. на Коши. Интегрален критерий на Коши

Д1 Редът от вида $\sum_{n=1}^{\infty} a_n$, където $a_n \geq 0$ ($\forall n \in N$) се нарича ред с неотрицателни членове.

Т1 Редът с неотр. членове $\sum_{n=1}^{\infty} a_n$ е сх $\Leftrightarrow \{S_n\}_{n=1}^{\infty}$ е огр.

Д-во:

Нека $S_n = \sum_{k=1}^{n} a_k \Rightarrow S_{n+1} - S_n = a_{n+1} \geq 0 \Rightarrow$

$S_{n+1} \geq S_n$, т.е. $\{S_n\}_{n=1}^{\infty}$ е мон. ↑ ред.

Тогава д.т.р. $\sum_{n=1}^{\infty} a_n$ е сх. $\Leftrightarrow \{S_n\}_{n=1}^{\infty}$ е сх. $\Leftrightarrow \{S_n\}_{n=1}^{\infty}$ е огр.

Т1 (Признак за сравн.) Нека д.т.р. $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ удовл. усл. $0 \leq a_n \leq b_n$ ($\forall n \in N$). Тогава:

1) ако $\sum_{n=1}^{\infty} b_n$ е сх $\Rightarrow \sum_{n=1}^{\infty} a_n$ е сх

2) ако $\sum_{n=1}^{\infty} a_n$ е разх $\Rightarrow \sum_{n=1}^{\infty} b_n$ е разх (следва от 1))

Д-во:

1) Нека $S_n = \sum_{k=1}^{n} a_k$, $S_n' = \sum_{k=1}^{n} b_k$

т.к. $\forall k = \overline{1,n}$, $a_n \le b_k \Rightarrow \underbrace{\sum_{k=1}^{n} a_k}_{S_n} \le \sum_{k=1}^{n} b_k = S_n'$

Редът $\sum_{n=1}^{\infty} b_n$ е сх $\Rightarrow$ ред $\{S_n'\}$ е огр, т.е.

$\exists M > 0 : \forall n \in \mathbb{N} : S_n' \le M$

От др. страна $\forall n \in \mathbb{N} : 0 \le S_n \le S_n' \le M \Rightarrow$

$\{S_n\}_{n=1}^{\infty}$ е огр $\overset{\text{Кр}}{\Rightarrow} \sum_{n=1}^{\infty} a_n$ е сх

Зад. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ - разх

$\sqrt{n} < n \quad \frac{1}{n} < \frac{1}{\sqrt{n}} \quad (\forall n \in \mathbb{N}$

Деф. Б.ч.р. $\sum_{n=1}^{\infty} a_n$ с $a_n \ge 0$

$(\forall n \in \mathbb{N})$ б.ч.р. с неотр числа

Тв. Б.ч.р ред $\sum_{n=1}^{\infty} a_n$, $a_n \ge 0$ е сх $\Leftrightarrow \{S_n\}_{n=1}^{\infty}$ е огр

Тв. Ако $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ т.че $0 \le a_n \le b_n$ $(\forall n \in \mathbb{N})$

$\Rightarrow$ 1) ако е сх. $\sum_{n=1}^{\infty} b_n \Rightarrow$ сх. $\sum_{n=1}^{\infty} a_n$ е сх.

2) ако е разх. $\sum_{n=1}^{\infty} a_n \Rightarrow$ разх $\sum_{n=1}^{\infty} b_n$

Пример. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ разх. $- \sum_{n=1}^{\infty} \frac{1}{n}$ - разх

1) $\sqrt{n} < n \Leftrightarrow 0 < \frac{1}{n} < \frac{1}{\sqrt{n}}$

2) Разгл. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} \underset{\text{сх}}{\Leftarrow} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=2}^{\infty} \frac{1}{n^2}$

$0 < \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ - сх.

$\Rightarrow$ сх. по призн. за сравн.

Критерий на Даламбер. Нека $\sum_{n=1}^{\infty} a_n : a_n > 0$ $(\forall n \in \mathbb{N})$

Ако: 1) $\exists\, 0 < q < 1 : \forall n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} \le q \Rightarrow \sum_{n=1}^{\infty} a_n$ е сх.

2) $\forall n \in \mathbb{N}$: $\frac{a_{n+1}}{a_n} \ge 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е разх.

$\left\{\frac{a_{n+1}}{a_n}\right\}_{n=1}^{\infty}$

Д-во:

1) Нека $\exists\, 0 < q < 1 : \forall n \in \mathbb{N} \rightarrow \frac{a_{n+1}}{a_n} \le q$

$-\frac{a_2}{a_1} \le q \Rightarrow a_2 \le a_1 q$

$\frac{a_3}{a_2} \le q \Rightarrow a_3 \le a_2 q \le a_1 q^2$

$\frac{a_4}{a_5} \le q \Rightarrow a_4 \le a_3 q \le a_2 q^2 \le a_1 q^3$

$\frac{a_{n+1}}{a_n} \le q \Rightarrow a_{n+1} \le a_n . q \le a_1 . q^n$

$-\sum_{n=0}^{\infty} a_1 q^n -$ сх? $= a_1 \sum_{n=0}^{\infty} q^n -$ сх, к. $|q| < 1$

$a_1 \le a_1 q^0 = a_1$

$\sum_{n=1}^{\infty} a_n$

$0 < a_n \le a_1 q^{n-1} (\forall n \in N)$

пр. за сравн. $\sum_{n=1}^{\infty} a_n$ е сх.

2) Нека $\forall n \in N: \frac{a_{n+1}}{a_n} \ge 1 \Rightarrow a_{n+1} \ge a_n (\forall n \in N)$

$0 < a_1 \le a_2 \le a_3 \dots$

$\sum_{n=1}^{\infty} a_1 -$ разх. $= \sum_{n=1}^{\infty} b_n$, $b_n = a_1 (\forall n \in N) \Rightarrow b_n = a_1 \to a_1 \neq 0$
$\hookrightarrow n^{ти}$ член не кл. към $0 \Rightarrow$ разх.

$\sum_{n=1}^{\infty} a_n \underset{\text{пр. за сравн.}}{\overset{\text{разх.}}{\Rightarrow}} \forall n \in N: 0 < a_1 \le a_n$

~~$\sum_{n=1}^{\infty} a_n$~~ Зад 1 Нека $\sum_{n=1}^{\infty} a_n, a_n > 0, (\forall n \in N)$
($\Leftrightarrow$ на кр. на Д.)

Ако: 1) $\exists\, 0 < q < 1, \exists N: \forall n > N \Rightarrow \frac{a_{n+1}}{a_n} \le q \Rightarrow$
$\sum_{n=1}^{\infty} a_n$ е сх.

2) $\exists N: \forall n > N \Rightarrow \frac{a_{n+1}}{a_n} \ge 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е разх.

Следствие | Нека $\sum_{n=1}^{\infty} a_n$, $a_n > 0$, $\forall n \in N$ и

$\exists \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \Rightarrow$ 1) Ако $l < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е сх

2) Ако $l > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е разх

3) Ако $l = 1$ - неопределеност

Д-во:

1) Нека $l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

$l-\varepsilon_0 \quad l \quad l+\varepsilon_0 \quad 1$

Избиране $\varepsilon_0 > 0$: $l + \varepsilon_0 < 1 \Rightarrow \varepsilon_0 > 0, \exists N = N(\varepsilon_0): \forall n > N \Rightarrow$

$\left| l - \frac{a_{n+1}}{a_n} \right| < \varepsilon_0 \Rightarrow$

$\cancel{l-\varepsilon_0 <} \frac{a_{n+1}}{a_n} < \underbrace{l + \varepsilon_0}_{q} < 1$

$\Rightarrow \exists q \in (0,1): \exists N: \forall n > N \Rightarrow \frac{a_{n+1}}{a_n} \le q \Rightarrow \sum_{n=1}^{\infty} a_n$ е сх.

2) Нека $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l > 1$

$1 \quad l-\varepsilon_0 \quad l \quad l+\varepsilon_0$

изб. $\varepsilon_0 > 0$: $1 < l - \varepsilon_0$

$\Rightarrow \varepsilon_0 > 0; \exists N: \forall n > N \Rightarrow \left| l - \frac{a_{n+1}}{a_n} \right| < \varepsilon_0 \Rightarrow 1 < \underbrace{l - \varepsilon_0}_{q} < \frac{a_{n+1}}{a_n} < \cancel{l + \varepsilon_0}$

$\Rightarrow \exists N: \forall n > N \Rightarrow \frac{a_{n+1}}{a_n} > q > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е разх.

Примери: 1) $\sum_{n=0}^{\infty} \frac{a^n}{n!}$, $a>0$ $\qquad a=1 \Rightarrow \sum_{n=0} \frac{1}{n!}$ е сх $\quad \| \; e=2,41$

$a_n = \frac{a^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{a_{n+1}}{(n+1)!}}{\frac{a_n}{n!}} = \frac{a}{n+1} \xrightarrow[n\to\infty]{} 0 = l < 1$$

2) $\sum_{n=1}^{\infty} \frac{1}{n}$ – разх, $a_n = \frac{1}{n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = 1 - \frac{1}{n+1} \xrightarrow[n\to\infty]{} 1 = l$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ е сх. $a_n = \frac{1}{n^2} > 0$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\left(\frac{1}{n}\right)^2} = \left(\frac{n}{n+1}\right)^2 = \left(1 - \frac{1}{n+1}\right)^2 \xrightarrow[n\to\infty]{} 1 = l$$

Т-критерий на Коши | Нека $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$, $\forall n \in \mathbb{N}$

Ако 1) $\exists\, 0 < q < 1 : \forall n > 1, \sqrt[n]{a_n} \leq q \Rightarrow \sum_{n=1}^{\infty} a_n$ е сх. $\qquad \{\sqrt[n]{a_n}\}_{n=2}^{\infty}$

2) $\forall n > 1 : \sqrt[n]{a_n} \geq 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е разх.

Забележка | Нека $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$ $(\forall n \in \mathbb{N})$

Ако: 1) $\exists\, 0 < q < 1$, $\exists N : \forall n > N \to \sqrt[n]{a_n} \leq q \Rightarrow$ $\sum_{n=1}^{\infty} a_n$ е сх.

2) $\exists N : \forall n > N \to \sqrt[n]{a_n} \geq 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е разх

Следствие | Нека $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$, $\forall n \in \mathbb{N}$ и $\exists \lim_{n\to\infty} \sqrt[n]{a_n} = l \Rightarrow$

1) ако $l < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е сх

2) ако $l > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ е разх.

3) $l = 1$ – неопр.

Пример: | $\binom{2n}{n} = \frac{2n!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2}$ $\qquad \sum_{n=1}^{\infty} \binom{2n}{n}$, $a_n = \binom{2n}{n} > 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\binom{2(n+1)}{n+1}}{\binom{2n}{n}} = \frac{\frac{[2(n+1)]!}{[(n+1)!]^2}}{\frac{2n!}{(n!)^2}} = \frac{[2(n+1)]!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2} =$$

$$= \frac{2(n+1)(2n+1)}{(n+1)^2} = 2\left(\frac{2n+1}{n+1}\right) = 2\left(2 - \frac{1}{n+1}\right) \xrightarrow[n\to\infty]{} 4 = l > 1$$

$\Rightarrow \sum_{n=1}^{\infty} \binom{2n}{n}$ е разх // $\lim_{n\to\infty} \binom{2n}{n} \neq 0$

Док-во:

1) Нека $\exists\, 0 < q < 1 : \forall n > 1 : \sqrt[n]{a_n} \leq q \Rightarrow$

1) $a_n \leq q^n$, $\sum_{n=2}^{\infty} a_n$, $\sum_{n=2}^{\infty} q^n = q^2 \sum_{n=2}^{\infty} q^{n-2}$ сх, $0 < q < 1$
сх. за ср.

$\forall n \geq 2 : 0 \leq a_n \leq q^n$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ е сх.

2) Нека $\forall n > 1 : \sqrt[n]{a_n} \geq 1 \Rightarrow a_n \geq 1$ $\qquad \sum_{n=2}^{\infty} a_n$, $\sum_{n=2}^{\infty} 1$ – разх

$a_n \geq 1 \geq 0$

$\Rightarrow$ (т-н за сравн.) $\sum_{n=2} a_n$ е разх $\Rightarrow \sum_{n=1} a_n$ е разх, $\lim a_n \neq 0$

Д-во (следствие):

1) $\lim_{n\to\infty} \sqrt[n]{a_n} = l$, $l < 1$

за $\varepsilon_0$, $\exists N: \forall n > N \Rightarrow |l - \sqrt[n]{a_n}| < \varepsilon_0$

$\Rightarrow l - \varepsilon_0 < \sqrt[n]{a_n} < \underbrace{l + \varepsilon_0}_{q} < 1 \xRightarrow{\text{кр.к.}} \sum_{n=1}^{\infty} a_n$ е сх.

2) $1 < l$ — избл. $\varepsilon_0 > 0$: $l - \varepsilon_0 > 1$

$\varepsilon_0 > 0$, $\exists N: \forall n > N: |l - \sqrt[n]{a_n}| < \varepsilon_0$

$1 - l - \varepsilon_0 < \sqrt[n]{a_n} < \cancel{l + \varepsilon_0}$

$\forall n > N: \sqrt[n]{a_n} > 1 \xRightarrow{\text{кр.к.}} \sum_{n=1}^{\infty} a_n$ е разх.

Пример: $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^{n^2}$, $a_n = \left(\frac{n+1}{n+2}\right)^{n^2} > 0$

$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n+1}{n+2}\right)^{n^2}} = \left(\frac{n+1}{n+2}\right)^n = \frac{(n+1)^n}{(n+2)^n} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n\left(1+\frac{2}{n}\right)^n} \longrightarrow \frac{e}{e^2} = \frac{1}{e} < 1$

$\Rightarrow$ (гр. сл. на кр. на К.) е сх.

Т-Интегрален критерий на Коши (Нека $f(x)$ е мон. $\downarrow$ в/у $[1, +\infty)$)

Б.т.р. $\sum_{n=1}^{\infty} f(n)$ е сх $\Leftrightarrow \int_1^{+\infty} f(x)\,dx$ е сх.

Д-во:

$\forall k \in \mathbb{N} \Rightarrow \forall x \in [k, k+1]: f(k+1) \leq f(x) \leq f(k)$ (м. $\downarrow$)

$\int_k^{k+1} f(k+1)\,dx \leq \int_k^{k+1} f(x)\,dx \leq \int_k^{k+1} f(k)\,dx$

$f(k+1) \leq \int_k^{k+1} f(x)\,dx \leq f(k)$ (за $\forall k \in \mathbb{N}$)

Нека $S_n = \sum_{k=1}^{n} f(k)$

$\sum_{k=1}^{n} f(k+1) \leq \sum_{k=1}^{n} \int_k^{k+1} f(x)\,dx \leq \sum_{k=1}^{n} f(k) = S_n$

$S_n \geq \int_1^{n+1} f(x)\,dx \geq S_{n+1} - f(1)$

$\sum_{s=2}^{n+1} f(s) = S_{n+1} - f(1)$

$\forall n \in \mathbb{N}: S_{n+1} - f(1) \leq \int_1^{n+1} f(x)\,dx$ ~~е огр. в/у $[1,+\infty)$~~ $\leq S_n$ (*)

$\Leftarrow$) Нека $\int_1^{+\infty} f(x)\,dx$ е сх $\Rightarrow$ ф-та $\int_1^{\xi} f(x)\,dx$ е огр в/у $[1, +\infty)$,

т.е. $\exists M > 0: \int_1^{\xi} f(x)\,dx \leq M$, $\forall \xi \geq 1$

$S_{n+1} \leq f(1) + \int_1^{n+1} f(x)\,dx \overset{(1)}{\leq} f(1) + M$ ($\forall n \in \mathbb{N}$) $\Rightarrow$

$\{S_{n+1}\}_{n=1}^{\infty}$ е огр. $\Rightarrow \{S_n\}_{n=1}^{\infty}$ е огр $\Rightarrow \sum_{n=1}^{\infty} f(n)$ е сх.

$\Rightarrow$) Нека $\sum_{n=1}^{\infty} f(n)$ е сх $\Rightarrow \{S_n\}_{n=1}^{\infty}$ е огр

$\exists M > 0: S_n \leq M$ ($\forall n \in \mathbb{N}$) $\xRightarrow{*} \forall n \in \mathbb{N}: \int_1^{n+1} f(x)\,dx \leq S_n \leq M$

$\forall \xi \in [1, +\infty) \Rightarrow \exists\ n:\ n \le \xi < n+1 \Rightarrow$

$$F(\xi) = \int_1^{\xi} f(x)\,dx \le \int_1^{n+1} f(x)\,dx \qquad \int_1^{\xi} + \int_{\xi}^{n+1} = \int_1^{n+1}$$

$\Rightarrow F(\xi)$ е огр. в/у $[1, +\infty) \Rightarrow \int_1^{+\infty} f(x)$ е сх.

Пример: $\sum_{n=1}^{\infty} \frac{1}{n^\alpha},\ (\alpha \in \mathbb{R}) \Rightarrow \begin{cases} \text{е сх., ако } \alpha > 1 \\ \text{е разх., ако } \alpha \le 1 \end{cases}$

1) $\alpha = 0\ \sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1$ – разх.

2) $\alpha < 0 \Rightarrow \frac{1}{n^\alpha} = n^{-\alpha} \xrightarrow[n\to\infty]{} +\infty$ – разх., т.к. $\lim_{n\to\infty} \frac{1}{n^\alpha} = +\infty$

• $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \text{сх}, & \alpha > 1 \\ \text{разх}, & 0 < \alpha \le 1 \end{cases}$

$f(x) = \frac{1}{x^\alpha} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, к. $f(x) = \frac{1}{x^\alpha}$

• $f'(x) = \left(\frac{1}{x^\alpha}\right)' = (x^{-\alpha})' = -\alpha \cdot x^{-\alpha-1} = \frac{-\alpha}{x^{\alpha+1}} < 0 \Rightarrow$

$f(x) = \frac{1}{x^\alpha}$ е м. непр. в/у $[1, +\infty)$

3) ако $0 < \alpha \le 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ е разх.

4) ако $\alpha > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ е сх.

## 15) Критерий на Лайбниц за редове с алтернативно сменящи се знаци

Деф. Редът от вида

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n,\ (a_n \ge 0,\ \forall n \in \mathbb{N})$$

се нарича ред с алт. сменящи се знаци.

$$\sum_{n=1}^{\infty} (-1)^{n} a_n = a_1 - a_2 + a_3 - a_4 \ldots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

Т (Критерий на Лайбниц)

Нека за реда $\sum_{n=1}^{\infty} (-1)^{n-1} a_n\ (a_n \ge 0,\ \forall n \in \mathbb{N})$ имаме:

1) $a_1 \ge a_2 \ge \ldots \ge a_n \ldots$

2) $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ е сх.

• $a_n = \frac{1}{n}$ – сх.

Д-во:

$S_n = \sum_{n=1}^{n} (-1)^{k-1} a_k$

разгл. подр. $S_2, S_4, S_6, \dots, S_{2n}, \dots$

1) $S_{2n} \le S_{2n+2}$

$S_{2n+2} = S_{2n} + \underbrace{a_{2n-1} - a_{2n+2}}_{\text{нех. инд. } \oplus} \ge S_{2n}$, т.е. ред. е н.р.

2) $S_{2n} = a_1 - a_2 + a_3 - a_4 + \dots + a_{2n-1} - a_{2n} \ge 0$ $\quad a_1$

$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}$

$\Rightarrow \forall n: 0 \le S_{2n} \le a_1 \overset{1,2}{\Rightarrow}$ т.е. $\{S_{2n}\}_{n=1}^{\infty}$ е огр.

Нека $S = \lim_{n\to\infty} S_{2n}$

Разгл. $S_1, S_3, S_5, \dots, S_{2n+1}, \dots$

$S_{2n+1} = S_{2n} + a_{2n+1}$

$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = S + 0 = S$

• $S_1, S_2, \dots, S_n, \dots \Rightarrow \exists \lim_{n\to\infty} S_n = S \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ е сх.

## (16) Условно и абсолютно сходящи се редове

Деф. 1) Б.ч.р. $\sum_{n=1}^{\infty} a_n$ се нарича абс. сх., ако е сх. $\sum_{n=1}^{\infty} |a_n|$

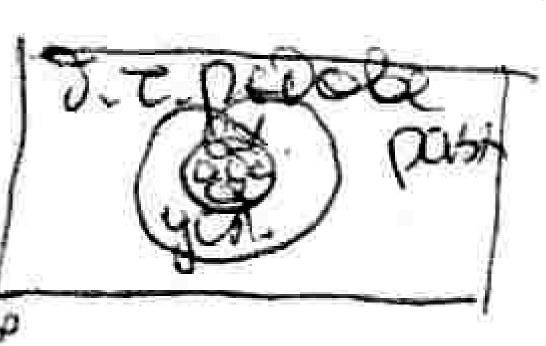
2) Б.ч.р. $\sum_{n=1}^{\infty} a_n$ се нар. усл. сх., ако е сх. и не е абс. сх.

• $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ - сх., $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \to$ разх.

усл. сх. $\qquad$ харм. ред

• $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, $a_n = \frac{1}{n^2} \to 0$, $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ сх.

сх. кр. Л.




Тh Ако б.ч.р. $\sum_{n=1}^{\infty} a_n$ е абс. сх. $\Rightarrow \sum_{n=1}^{\infty} a_n$ е сх.

Д-во:

$\sum_{n=1}^{\infty} a_n$ е абс. сх. $\Rightarrow$ е сх. $\sum_{n=1}^{\infty} |a_n| \Rightarrow$ (Кр. Коши)

$\forall \varepsilon > 0, \exists N = N_\varepsilon: \forall n > N_\varepsilon, \forall p \in \mathbb{N} \Rightarrow \left|\sum_{k=1}^{p} a_{n+k}\right| < \varepsilon$?

т.к. $\left|\sum_{k=1}^{p} a_{n+k}\right| \le \sum_{k=1}^{p} |a_{n+k}| < \varepsilon \Rightarrow$ (Кр. Коши) $\sum_{n=1}^{\infty} a_n$ е сходящ

Пример: 1) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \to$ абс. сх., т.к. $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ е сх.

2) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ е сх. по кр. на Лайбниц, но $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ - х.р. $\to$ разходящ

$\boxed{T}$ Нека $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ са абс. сх. $\Rightarrow$

1) $\sum_{n=1}^{\infty} (a_n + b_n)$ е абс сх. ;

2) $\sum_{n=1}^{\infty} \lambda a_n$ $(\lambda \in \mathbb{R})$ е абс. сх.

Д-во:

1) $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ - абс. сх., т.е. са сх. $\sum_{n=1}^{\infty} |a_n|$ и $\sum_{n=1}^{\infty} |b_n| \Rightarrow$

сх. е $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$, т.к. за $\forall\, n \in \mathbb{N}$:

$|a_n + b_n| \leq |a_n| + |b_n| \Rightarrow$ (п-п зафравняването)

$\sum_{n=1}^{\infty} |a_n + b_n|$ е сх $\xRightarrow{D.}$ $\sum_{n=1}^{\infty} (a_n + b_n)$ е сх.

2) $\sum_{n=1}^{\infty} a_n$ е абс. сх $\Rightarrow \sum_{n=1}^{\infty} |a_n|$ е сх. $\Rightarrow |\lambda| \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |\lambda a_n|$ е сх $\Rightarrow$

$\sum_{n=1}^{\infty} \lambda a_n$ е абс. сх, к. $(\lambda \in \mathbb{R})$

$\boxed{T}$ Ако д.ч. ред $\sum_{n=1}^{\infty} a_n$ е абс. сх. $\Rightarrow$

д.ч.р. $\sum_{n=1}^{\infty} a_{\pi(n)}$, където $\pi: \mathbb{N} \to \mathbb{N}$ е биекция,

е абс. сх. и $\sum_{n=1}^{\infty} a_{\pi(n)} = S$ $(S = \sum_{n=1}^{\infty} a_n)$

Изображението $\pi$ е биекция, ако:

1) $\forall\, n \in \mathbb{N}, \exists\, m_n \in \mathbb{N}: \pi(m_n) = n$

2) $\forall\, n, m \in \mathbb{N}, n \neq m \Rightarrow \pi(n) \neq \pi(m)$

Д-во:

$S_n = \sum_{k=1}^{n} a_k$, $S_n^{|\cdot|} = \sum_{k=1}^{n} |a_k|$, $S_n(\pi) = \sum_{k=1}^{n} a_{\pi(k)}$, $S_n^{|\cdot|}(\pi) = \sum_{k=1}^{n} |a_{\pi(k)}|$

1) $\sum_{n=1}^{\infty} a_{\pi(n)}$ е абс. сх, т.е. $\sum_{n=1}^{\infty} |a_{\pi(n)}|$ е сх?

т.к. $\sum_{n=1}^{\infty} a_n$ е абс. сх $\Rightarrow$ сх. е $\sum_{n=1}^{\infty} |a_n| \Rightarrow \{S_n^{|\cdot|}\}_{n=1}^{\infty}$ е огр. $\Rightarrow$

$\exists\, M > 0: 0 \leq S_n^{|\cdot|} \leq M$ $(*)$

Взимаме $S_n^{|\cdot|}(\pi) = \sum_{k=1}^{n} |a_{\pi(k)}|$, нека $m_n = \max_{1 \leq k \leq n} \pi(k) \Rightarrow$

$S_n^{|\cdot|}(\pi) = \sum_{k=1}^{n} |a_{\pi(k)}| \leq \sum_{k=1}^{m_n} |a_k| = S_{m_n}^{|\cdot|} \overset{(*)}{\leq} M$

$\Rightarrow \{S_n^{|\cdot|}(\pi)\}_{n=1}^{\infty}$ е огр $\Rightarrow \sum_{n=1}^{\infty} |a_{\pi(n)}|$ е сх.

$\Rightarrow \sum_{n=1}^{\infty} a_{\pi(n)}$ е абс. сходящ

$\boxed{T2}$ $\sum_{n=1}^{\infty} a_{\pi(n)}$ е сх $\Rightarrow \sum_{n=1}^{\infty} a_{\pi(n)} = S$, където $S = \sum_{n=1}^{\infty} a_n$ ?

Д-во:

Т.к. $\sum_{n=1} a_n$ е абс. сх. $\Rightarrow \sum_{n=1} (a_n)$ и $\sum_{n=1} a_n = S$ - сходящ

(? $\lim_{n\to\infty} S_n(\pi) = S \Leftrightarrow \forall \varepsilon > 0, \exists \nu: \forall n > \nu \Rightarrow |S - S_n(\pi)| < \varepsilon$ ?)

$\forall \varepsilon > 0 \to \exists \nu_1, \forall n > \nu_1, \forall p \in \mathbb{N} \Rightarrow \sum_{k=1}^{p} |a_{n+k}| < \frac{\varepsilon}{2},$

$\exists \nu_2, \forall n > \nu_2 \Rightarrow |S - S_n| < \frac{\varepsilon}{2}$

Взимаме $\nu > \max\{\nu_1, \nu_2\} \Rightarrow$

$\forall n > \nu_1, \forall p \in \mathbb{N} \Rightarrow \begin{cases} \sum_{k=1}^{p} |a_{n+k}| < \frac{\varepsilon}{2} \ (1) \\ |S - S_n| < \frac{\varepsilon}{2} \ (2) \end{cases}$

$\forall k \in \mathbb{N}, k \leq \nu, \exists ! m_k \in \mathbb{N}: \pi(m_k) = k$, т.е. $\pi^{-1}(k) = m_k$

Нека $\bar{\nu} = \max_{1 \leq k \leq \nu} m_k \ (\bar{\nu} \geq \nu)$

$\{1, 2, 3, \dots, \nu\}$
$\{m_1, m_2, \dots, m_\nu\}$
$\bar{\nu}$ - гр. сл. $\{1, \dots, \bar{\nu}\}$
$\{1, 2, 3, \dots \bar{\nu}\} \supset \{m_1, \dots, m_\nu\}$

$\forall n > \bar{\nu} \Rightarrow |S - S_n(\pi)| = |(S - S_n) + (S_n - S_n(\pi))| \leq$

$\leq |S - S_n| + |S_n(\pi) - S_n| \overset{(2)}{<} \frac{\varepsilon}{2} + |S_n(\pi) - S_n| = (*)$

$S_n(\pi) - S_n = \sum_{n=1}^{\nu} a_k = \sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{\nu} a_{\pi(m_k)} = \sum_{k \in M} a_{\pi(k)},$

$\{1, 2, \dots, n\} \supset \{m_1, \dots, m_\nu\}$

където $M = \{1, 2, \dots, n\} \setminus \{m_1, m_2, \dots, m_\nu\}$ (3)

$* \overset{(3)}{=} \frac{\varepsilon}{2} + |\sum_{k \in M} a_{\pi(k)}| \leq \frac{\varepsilon}{2} + \sum_{k \in M} |a_{\pi(k)}| = (**)$

Нека $\hat{\nu} = \max_{k \in M} \pi(k) \Rightarrow \exists p \in \mathbb{N}: \nu + p = \max_{k \in M} \pi(k) = \hat{\nu}$

$p = \hat{\nu} - \nu$, т.к. $\forall k \in M: \pi(k) > \nu$, $\nu + 1 \leq \pi(k) \leq \nu + p$

$\{\pi(k): k \in M\} \subset \{\nu+1, \dots, \nu+p\}$

$(**) = \frac{\varepsilon}{2} + \sum_{k \in M} |a_{\pi(k)}| \leq \frac{\varepsilon}{2} + \sum_{k=1}^{p} |a_{\nu+k}| \overset{(1)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \lim_{n\to\infty} S_n(\pi) = S$, т.е. $\sum_{n=1}^{\infty} a_{\pi(n)} = S = \sum_{n=1}^{\infty} a_n$

<u>Т (Риман)</u> Ако ч.р.ред $\sum_{n=1}^{\infty} a_n$ е усл. сх. $\Rightarrow$

$\forall L \in \mathbb{R} \cup \{\pm\infty\}, \exists$ биекция $\pi: \mathbb{N} \to \mathbb{N}: \sum_{n=1}^{\infty} a_{\pi(n)} = L$

<u>Т</u> Нека $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ са абс. сх. редове $\Rightarrow$

ч.р.р. $\sum_{n,m \in \mathbb{N}} a_n b_m$, получен от сумирането на $\{a_n, b_m\}_{n,m \in \mathbb{N}}$ в някакъв ред е абс. сходящ.

Неговата сума е равна на произв. на $S, \bar{S}$, където

$S = \sum_{n=1}^{\infty} a_n$, $\bar{S} = \sum_{n=1}^{\infty} b_n$

| | | | |
|---|---|---|---|
| $a_1 b_1$ | $a_1 b_2$ | $a_1 b_3$ | — |
| $a_2 b_1$ | $a_2 b_2$ | $a_2 b_3$ | — |
| $a_3 b_1$ | $a_3 b_2$ | $a_3 b_3$ | — |

$* \ a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2 + a_1 b_3 + a_3 b_1 + a_2 b_3 + a_3 b_2 + a_3 b_3 + \dots$

$\hat{S}_n$ - $n^{\text{та}}$ парц. сума на (*)

$S_n^{2\wedge} = S_n . \overline{S_n}$, където $S_n = \sum_{k=1}^{n} a_k$, $\overline{S_n} = \sum_{k=1}^{n} b_k$

• |(*)| $|a_1 b_1| + |a_1 b_2| + |a_2 b_1| + |a_2 b_2| + \dots = \hat{S}^{(\cdot)}$ - $n^{\text{та}}$ парц. сума на |(*)|

$\Rightarrow \hat{S}_{n^2}^{(\cdot)} = S_n^{(\cdot)} . \overline{S_n}^{(\cdot)}$, к. $S_n^{(\cdot)} = \sum_{k=1}^{n} |a_k|$, $\overline{S_n}^{(\cdot)} = \sum_{k=1}^{n} |b_k|$

т.к. $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ са абс. сх. $\Rightarrow$

$\{S_n^{(\cdot)}\}_{n=1}^{\infty}$ и $\{\overline{S_n}^{(\cdot)}\}_{n=1}^{\infty}$ са огр., т.е. $\exists M > 0$: $\begin{cases} S_n^{(\cdot)} \le M \\ \overline{S}_n^{(\cdot)} \le M \end{cases}$

$\Rightarrow \hat{S}_{n^2}^{(\cdot)} = S_n^{(\cdot)} . \overline{S_n}^{(\cdot)} \le M.M = M^2 \Rightarrow \{\hat{S}_{n^2}^{(\cdot)}\}_{n=1}^{\infty}$ е огр. $\Rightarrow$

$\hat{S}_n^{(\cdot)} \le \hat{S}_{n^2}^{(\cdot)} \le M^2 \Rightarrow \{\hat{S}_n^{(\cdot)}\}_{n=1}^{\infty}$ е огр. $\Rightarrow$

|(*)| е сх. $\Rightarrow$ (*) е абс. сх.

$\hat{S}_{n^2} = S_n . \overline{S_n} \xrightarrow[n\to\infty]{} S.\overline{S} \Rightarrow$ сумата на (*) е $S.\overline{S}$

$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n b_m = S.\overline{S} = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_m\right)$

## 17. Функционални редици и редове - сходимост и равномерна сходимост. Критерий на Вайерщрас

**Деф.** Нека $\{f_n(x)\}_{n=1}^{\infty}$ е деф. в/у $E \subset \mathbb{R}$
Нека $x_0 \in E$.

1) Казваме, че д.ф.р. $\{f_n(x)\}_{n=1}^{\infty}$ е сх. в т. $x_0$, ако д.ч.р. $\{f_n(x_0)\}_{n=1}^{\infty}$ е сх.

2) Казваме, че д. функционална редица $\{f_n(x)\}_{n=1}^{\infty}$ е сх. в/у $E$, ако $\forall x \in E$, д.ч.р. $\{f_n(x)\}_{n=1}^{\infty}$ е сх.

3) Казваме, че д.ф.р. $\{f_n(x)\}_{n=1}^{\infty}$ е сх към ф-та $f(x)$ в/у $E$, ако $\forall x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$ : $f_n(x) \xrightarrow[n\to\infty]{E} f(x)$

**Пример:** 2) $\forall n \in \mathbb{N}: f_n(x) = x^{n-1}$, $x \in (-1,1)$, $\forall x \in (-1,1)$:

$1, x, x^2, \dots, x^{n-1}$ : $\lim_{n\to\infty} x^{n-1} = 0$ в/у $(-1,1)$

3) $f_n(x) = x^{n-1}$, $f(x) = 0$

$x^{n-1} \xrightarrow[n\to\infty]{(-1,1)} 0$

$x_0 = 0$, $f_n(0) = 0^n 0 = 0$ $\quad f_n(0) \xrightarrow[n\to\infty]{} 0$

$f_n(x) \xrightarrow[n\to\infty]{E} f(x) \Leftrightarrow \forall x \in E$, $f_n(x) \longrightarrow f(x) \Leftrightarrow$

$\forall x \in E, \forall \varepsilon > 0, \exists N = N(x,\varepsilon): \forall n > N \Rightarrow |f(x) - f_n(x)| < \varepsilon$

**Деф.** Казваме, че д.ф.р. $\{f_n(x)\}$ е равномерно сх. към $f(x)$ в/у $E$, ако $\forall \varepsilon > 0, \exists N = N(\varepsilon): \forall x \in E, \forall n > N \Rightarrow |f(x) - f_n(x)| < \varepsilon$ $\quad f_n(x) \underset{n\to\infty}{\overset{E}{\rightrightarrows}} f(x)$ равном. сх.

**Т.** $f_n(x) \underset{n\to\infty}{\overset{E}{\rightrightarrows}} 0 \Leftrightarrow \sup_{x \in E} |f_n(x)| \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$] Нека $f(x) \underset{n\to\infty}{\overset{E}{\rightrightarrows}} 0$, т.е. $\forall \varepsilon > 0, \exists N = N(\varepsilon), \forall n > N \Rightarrow$

$|f_n(x)| < \frac{\varepsilon}{2}, (\forall x \in E)$

$\Downarrow$

$\sup\limits_{x\in E} |f_n(x)| \leq \frac{\varepsilon}{2} < \varepsilon$

$\Rightarrow \lim\limits_{n\to\infty} \sup\limits_{x\in E} |f_n(x)|$

$\Leftarrow$] Нека $\sup\limits_{x\in E} |f_n(x)| \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon)$

$\forall n > N \Rightarrow |f_n(x)| \underset{x\in E}{\leq} \sup\limits_{x\in E} |f_n(x)| < \varepsilon$

$|f_n(x)| < \varepsilon, (\forall x \in E) \Rightarrow f_n(x) \underset{n\to\infty}{\overset{E}{\rightrightarrows}} 0$

Пример: 1) $f_n(x) = x^{n-1} \underset{n\to\infty}{\overset{(-1,1)}{\longrightarrow}} 0$

$x^{n-1} \underset{n\to\infty}{\overset{(-1,1)}{\not\rightrightarrows}} 0$ ? (равном. сх. ли е?)

$1 = \sup\limits_{x\in(-1,1)} |x^{n-1}| \underset{n\to\infty}{\not\longrightarrow} 0$

2) $x^{n-1} \underset{n\to\infty}{\overset{[-q,q]}{\rightrightarrows}} 0$, където $0 < q < 1$

$\sup\limits_{x\in[-q,q]} |x^{n-1}| = q^n \longrightarrow 0$

Следствие: $f_n(x) \overset{E}{\rightrightarrows} f(x) \Leftrightarrow \sup\limits_{x\in E} |f(x) - f_n(x)| \underset{n\to\infty}{\longrightarrow} 0$

Т] Ако $f_n(x) \underset{n\to\infty}{\overset{E}{\rightrightarrows}} f(x)$ и $F \subset E \Rightarrow$

$f_n(x) \underset{n\to\infty}{\overset{F}{\rightrightarrows}} f(x)$

$0 \leq \sup\limits_{x\in F} |f(x) - f_n(x)| \leq \sup\limits_{x\in E} |f(x) - f_n(x)|$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$0 \qquad 0 \qquad 0$

Пример: $x^{n-1} \underset{n\to\infty}{\overset{E}{\rightrightarrows}} 0$, където $[a,b] \subset (-1,1)$ 

$\exists\, 0 < q < 1 : [a,b] \subset [-q,q]$

Деф] Нека $f_n(x)$, $(n \in N)$, деф в/у $E \subset R$

$\sum\limits_{n=1}^{\infty} f_n(x)$ - функционален ред.

Деф] Казваме, че ф.р. $\sum\limits_{n=1}^{\infty} f_n(x)$, определена в/у $E$ е:

1) сх. в т. $x_0$, ако ч.р. $\sum\limits_{n=1}^{\infty} f_n(x)$ е сх.

2) сх. в/у мн. $E$, ако $\forall x \in E$, ч.р. $\sum\limits_{n=1}^{\infty} f_n(x)$ е сх.

Пример: $\sum\limits_{n=1}^{\infty} x^{n-1}$, $(-1,1)$ е сх.

3) Казваме, че $f(x)$ е сума на ф. ред $\sum\limits_{n=1}^{\infty} f_n(x)$ в/у $E$, ако редицата от парциалната сума $S_n(x) = \sum\limits_{n=1}^{n} f_k(x)$ е сх към $f(x)$, $\forall x \in E$

$f(x)=\sum_{n=1}^{\infty} f_n(x)\ (x\in E) \iff x\in E,$

$\forall \varepsilon>0, \exists \nu=\nu(x,\varepsilon): \forall n>\nu \Rightarrow |f(x)-S_n(x)|<\varepsilon$

Деф| Казваме, че ~~f(x) e~~ б. ф. р. $\sum_{n=1}^{\infty} f_n(x)$ е равном. сх. към $f(x)$ в/у м-вото E, ако $S_n(x) \overset{E}{\underset{n\to\infty}{\rightrightarrows}} f(x)$, т.е. $\forall \varepsilon>0, \exists \nu=\nu(\varepsilon): \forall n>\nu \Rightarrow$

$$|f(x)-S_n(x)|<\varepsilon\ (\forall x\in E)$$

Нека $\sum_{n=1}^{\infty} f_n(x)=f(x)$ в/у E. Означаваме чрез $r_n(x)=f(x)-S_n(x)$

Т| Функционалният ред $\sum_{n=1}^{\infty} f_n(x)$ е равномерно сх. в/у $E \iff$

$$\lim_{n\to\infty}\sup_{x\in E}|r_n(x)|$$

Д-во:

$\sum_{n=1}^{\infty} f_n(x)$ е равном. сх. $\iff S_n(x) \overset{E}{\underset{n\to\infty}{\rightrightarrows}} f(x) \iff$

$$\sup_{x\in E}|\underbrace{f(x)-S_n(x)}_{r_n(x)}| \xrightarrow[n\to\infty]{} 0 \iff \sup_{x\in E}|r_n(x)| \xrightarrow[n\to q]{} 0$$

Пример: $\sum_{n=1}^{\infty} x^{n-1}=\frac{1}{1-x}$; $x\in(-1,1)$

i) $E=[-q,q]$, $0<q<1$

$\sum_{n=1}^{\infty} x^{n-1}$ е равн. сх. в/у $[-q,q]$

$f(x)=\frac{1}{1-x}$, $S_n(x)=\sum_{k=1}^{n} x^{k-1}=\frac{1-x^n}{1-x}$

$\underbrace{f(x)-S_n(x)}_{r_n(x)}=\frac{1}{1-x}-\frac{1-x^n}{1-x}=\frac{x^n}{1-x}$

$|r_n(x)|=\left|\frac{x^n}{1-x}\right|=\frac{|x|^n}{|1-x|}\le\frac{|x^n|}{1-|x|}\le\frac{q^n}{1-q}$

$\sup_{x\in[-q,q]}|r_n(x)|=\sup_{x\in[-q,q]}\frac{|x|^n}{|1-x|}\le\frac{q^n}{1-q}\xrightarrow[n\to\infty]{}\frac{0}{1-q}=0$

$\Rightarrow \sum_{n=1}^{\infty} x^{n-1}$ е равн. сх. в/у $[-q,q]$

ii) $\sum_{n=1}^{\infty} x^{n-1}$ не е равн. сх. в/у $(-1,1)$

$\forall n\in\mathbb{N}: x_n=1-\frac{1}{n}$

$\sup_{x\in(-1,1)}|r_n(x)|=\sup_{x\in(-1,1)}\frac{|x|^n}{|1-x|}\overset{\forall}{\ge}|r_n(x_n)|=\frac{(1-\frac{1}{n})^n}{|1-(1-\frac{1}{n})|}=n(1-\frac{1}{n})^n$

$\sup_{x\in(-1,1)}|r_n(x)|\ge n(1-\frac{1}{n})^n\xrightarrow[n\to+\infty]{}+\infty$

$e^{-1}$ (~~конст.~~ $\cdot$ по (+)) $=+\infty$

$\Rightarrow \lim_{n\to\infty}\sup_{x\in(-1,1)}|r_n(x)|=+\infty\ne0$

$\sum_{n=1}^{\infty} x^{n-1}$ не е равном. сходящо в/у $(-1,1)$

**Т** /Критерий на Вайерщрас/

Ако за ф. ред $\sum_{n=1}^{\infty} f_n(x)$, $(x\in E)$, $\exists \sum_{n=1}^{\infty} a_n$, $(a_n\geq 0)$: $\forall n\in\mathbb{N}$:

$$|f_n(x)|\leq a_n,\ (\forall x\in E) \Rightarrow$$

$\sum_{n=1}^{\infty} f_n(x)$ е абс. равном. сх. в/у $E$

(редът от сума от модулите)

**Деф** Казваме, че $\sum_{n=1}^{\infty} f_n(x)$ е абс. сх. в/у $E$, ако е сх. в/у $E$ ф. ред, $\sum_{n=1}^{\infty} |f_n(x)|$

Д-во:

$$r_n(x)=\sum_{k=n+1}^{\infty} f_k(x)$$

$$\lim_{n\to\infty}\sup_{x\in E}|r_n(x)|=0\ ?$$

$r_n(x)$ — $\sum_{n=1}^{\infty}|f_n(x)|$

$$|r_n(x)|=\left|\sum_{k=n+1}^{\infty} f_k(x)\right|\leq \sum_{k=n+1}^{\infty}|f_k(x)|\leq\sum_{k=n+1}^{\infty} a_k$$

$$\sum_{n=1}^{\infty} a_n \text{ е сх} \Rightarrow \sum_{k=n+1}^{\infty} a_k=(S-S_n)\xrightarrow[n\to\infty]{} 0$$

$$\underset{\downarrow\, 0}{0}\leq \sup\underset{\downarrow\, 0}{|r_n(x)|}\leq \underset{\downarrow\, 0}{\sum_{k=n+1}^{\infty} a_k} \Rightarrow \sum_{n=1}^{\infty} f_n(x) \text{ е равн. сх. в/у } E$$

- $\sum_{n=1}^{\infty} x^{n-1}$, $E=[-q,q]$, $0<q<1$

  $\forall x\in[-q,q]: |x^n|\leq q^n$

  $\sum_{n=1}^{\infty} q^{n-1}$ е сх $\xRightarrow{\text{кр. В.}}$ $\sum_{n=1}^{\infty} x^{n-1}$ е абс. и равн. сх. в/у $[-q,q]$

## 18. Степенни редове - радиус и област на сходимост.

**Деф** Функц. ред от вида $(*)\ \sum_{n=0}^{\infty} a_n(x-x_0)^n$, където $a_n\in\mathbb{R}$, $\forall n\in\mathbb{N}\cup\{0\}$, $x_0\in\mathbb{R}$, се нарича <u>степенен ред</u>.

<u>В частност:</u> Ако $x_0=0 \Rightarrow \sum_{n=0}^{\infty} a_n x^n$

В т. $x=x_0$ степенният ред $(*)$ е сх.

$(1)\ \sum_{n=0}^{\infty} a_n(x-x_0)^n$ пол. $t=x-x_0$

$(2)\ \sum_{n=0}^{\infty} a_n t^n$

Ако (1) е сх. в т. $x \Rightarrow$ ст. ред (2) е сх. в т. $t=x-x_0$

Ако (2) е сх. в т. $t \Rightarrow$ ст. ред (1) е сх. в т. $x=t+x_0$

**Т 1 (Абел)** Ако ст. ред $\sum\limits_{n=0}^{\infty} a_n x^n$ е сх. в т. $x_0 \neq 0 \Rightarrow$ е сх. при това абс. сх. във вс. т. $x : |x| < |x_0|$

**Д-во:**

Ст. ред $\sum\limits_{n=1}^{\infty} a_n x^n$ е сх в т. $x_0$, т.е. е сх. ч.р. $\sum\limits_{n=0}^{\infty} a_n x_0^n \Rightarrow a_n x_0^n \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \{a_n x_0^n\}_{n=0}^{\infty}$ е огр. т.е. $\exists M > 0 : |a_n x_0^n| \le M \ (\forall n \in \mathbb{N} \cup \{0\})$

Разгл. $\sum\limits_{n=0}^{\infty} |a_n x^n|$

(*) $\forall n \in \mathbb{N} \cup \{0\} : |a_n x^n| = \left|a_n x_0^n \left(\frac{x^n}{x_0^n}\right)\right| = |a_n x_0^n| \cdot \left|\frac{x}{x_0}\right|^n \le M \left|\frac{x}{x_0}\right|^n = M q^n$, където $q = \left|\frac{x}{x_0}\right|$

$\sum\limits_{n=0}^{\infty} |a_n x^n|$ сх. & $\sum\limits_{n=0}^{\infty} M q^n$ е сх., ако $q < 1 \Leftrightarrow \left|\frac{x}{x_0}\right| < 1 \Leftrightarrow \forall |x| < |x_0|$

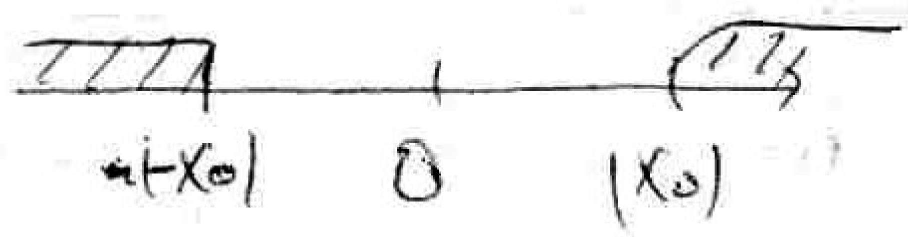
п-п за сравнение

$\forall n \in \mathbb{N} \cup \{0\}, \ |a_n x^n| \le M q^n \Rightarrow$

$\sum\limits_{n=0}^{\infty} |a_n x^n|$ е сх. $\forall x : |x| < |x_0| \Rightarrow \sum\limits_{n=0}^{\infty} a_n x^n$ е абс. сх. $\forall x : |x| < |x_0|$

$\Rightarrow \sum\limits_{n=0}^{\infty} a_n x^n$ е сх. $\forall x \ |x| < |x_0|$

**Следствие 1:** Ако ст. ред $\sum\limits_{n=0}^{\infty} a_n x^n$ е разх. в т. $x_0$ $(x_0 \neq 0) \Rightarrow$ ст. ред $\sum\limits_{n=0}^{\infty} a_n x^n$ е разх $\forall x : |x| > |x_0|$



**Следствие 2:** Ако ст. ред $\sum\limits_{n=0}^{\infty} a_n x^n$ е сх. в т. $x_0 \neq 0 \Rightarrow$ ст. ред $\sum\limits_{n=0}^{\infty} a_n x^n$ е равномерно сходящ в/у $[-\rho, \rho]$, к. $0 < \rho < |x_0|$

**Д-во:**

$|a_n x^n| = \left|a_n x_0^n \cdot \frac{x^n}{x_0^n}\right| = |a_n x_0^n| \cdot \left|\frac{x^n}{x_0^n}\right| \le M \left|\frac{x}{x_0}\right|^n \le M \left|\frac{\rho}{x_0}\right|^n, \ \forall x \in [-\rho, \rho]$

$\rho < |x_0|, \ 0 < \frac{\rho}{x_0} < 1$

Б.т.р. $\sum\limits_{n=0}^{\infty} M \left|\frac{\rho}{x_0}\right|^n$ е сх.

$\hookrightarrow$ безкр. геом. п-я

$\Rightarrow$ (кр. В.) ф. р. $\sum\limits_{n=0}^{\infty} a_n x^n$ е абсолютно (равном.) сходящ в/у $[-\rho, \rho]$

**Т за $\exists$ на радиус:** За $\forall$ степенен ред $\sum\limits_{n=0}^{\infty} a_n x^n$, $\exists! \ R \ge 0$ или $R = +\infty$, ако: 1) $R = 0 \Rightarrow$ ст. р. $\sum\limits_{n=0}^{\infty} a_n x^n$ е сх. само в т. $x_0 = 0$

2) $R = +\infty \Rightarrow$ ст. р. $\sum\limits_{n=0}^{\infty} a_n x^n$ е сх. в/у $(-\infty, +\infty)$

3) $0 < R < +\infty \Rightarrow$ ст. ред $\sum_{n=0} a_n x^n$ е сх. във $(-R, R)$ и разх във $(-\infty, -R) \cup (R, +\infty)$

Д-во:

Нека $D = \{x \in \mathbb{R} : $ ст. р. $\sum_{n=0}^{\infty} a_n x^n$ е сх.$\}$, $D \neq \emptyset$, т.к. $D \ni 0$

• Случай 1: Нека $D$ е неогр., т.е. $\forall x \in \mathbb{R}, \exists x_0 \in D : |x| < |x_0|$ $\xRightarrow{\text{т. Абел}}$ $\sum_{n=0}^{\infty} a_n x^n$ е сх. в т. $x \in \mathbb{R} \Rightarrow x \in D \Rightarrow D = (-\infty, +\infty) \Rightarrow R = +\infty$

• Случай 2: Нека $D$ е огр. м-во

2.1) $D = \{0\} \Rightarrow$ ст. ред $\sum_{n=0}^{\infty} a_n x^n$ е сх., само в т. $x_0 = 0$ и е разх. $\forall x \neq 0 \Rightarrow R = 0$

2.2) $D \neq \{0\} \Rightarrow \exists x \neq 0$: ст. р. $\sum_{n=0} a_n x^n$ е сх. в т. $X$

Нека $R = \sup_{x \in D} |x| > 0$

$\forall x \in (-R, R) \Leftrightarrow |x| < R \Rightarrow \exists x_0 \in D : |x| < |x_0| \xRightarrow{\text{т. Абел}}$ $\sum_{n=0}^{\infty} a_n x^n$ е сх. в тази т. $X$

$\forall x \in (-\infty, -R) \cup (R, +\infty)$ т.е. $|x| > R \Rightarrow x_0 \notin D$ : $|x| > |x_0| \xRightarrow{\text{сл. 1}} \sum_{n=0}^{\infty} a_n x^n$ е разх. в тази т. $X$

Деф $R$ - радиус на сх. на ст. ред $\sum_{n=0}^{\infty} a_n x^n$, $(-R, R)$ - област на ст. ред $\sum_{n=0}^{\infty} a_n x^n$

## 19. ~~Почленно диференциране и интегриране на степенни редове. Ред на Тейлор.~~ ~~Разлагане в ред на Тейлор на някои елементарни функции.~~

Т/ Нека за ст. ред (*) $\sum_{n=0}^{\infty} a_n x^n$ е в сила поне 1 от 2-те твърдения:

1) $\exists \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell$ или

2) $\exists \lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell$

$\Rightarrow R = \frac{1}{\ell}$ е радиус на сх. на (*)

Д-во:

2) Нека $\exists \lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell$

1) $0 < \ell < +\infty$

Разгл. $\sum_{n=0}^{\infty} |a_n x^n|$.

$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} |x| \sqrt[n]{|a_n|} = |x| \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x| \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x| \cdot \ell$

Ако $|x| \ell > 1 \Leftrightarrow |x| > \frac{1}{\ell} \Rightarrow \sum_{n=0}^{\infty} |a_n x^n|$ е разх

Ако $|x|\ell<1 \iff |x|<\frac{1}{\ell} \Rightarrow \sum_{n=0} a_n x^n$ е сх $(-\frac{1}{\ell}, \frac{1}{\ell})$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ е разх. Действ. ; ако доп., че $\sum_{n=0}^{\infty} a_n x^n$ е сх за някои $x$

$|x|>\frac{1}{\ell} \Rightarrow$ в т. $x_0: |x| > |x_0| > \frac{1}{\ell} \overset{\text{Т. Абел}}{\Rightarrow}$ ст. ред $\sum_{n=0}^{\infty} a_n x_0^n$ е абс. сх.

$|x_0|\ell > 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n x_0^n|} = |x|.\ell > 1$ ↯

$R=\frac{1}{\ell}$ е ред на сходимост на $(*)$

ii) $\ell=0$

$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x|.0 = 0 < 1 \Rightarrow \sum_{n=0}^{\infty} |a_n x^n|$ е сх. $\forall x \in R$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ е сх. $\forall x \in R$, т.е. $R = \infty = \frac{1}{\ell} = \frac{1}{0}$

iii) $\ell=+\infty$

$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x|.\lim_{n\to\infty} \sqrt[n]{|a_n|} = |x|(+\infty) = \begin{cases} +\infty, & x\neq 0 \\ 0, & x=0 \end{cases}$

$R = 0 = \frac{1}{\ell} = \frac{1}{+\infty}$

<u>Примери</u>:

1) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $a_n \frac{1}{n!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0 = \ell$

$\Rightarrow R = \frac{1}{0} = +\infty$

2) $\sum_{n=1}^{\infty} \frac{x^n}{n.3^n}$, $a_n = \frac{1}{n.3^n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n.3^n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \cdot \frac{1}{3} = \frac{1}{3} \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \frac{1}{3}\cdot\frac{1}{1} = \frac{1}{3} = \ell$

$\Rightarrow R = \frac{1}{\ell} = \frac{1}{\frac{1}{3}} = 3$

<u>Следствие</u>: (1) $\sum_{n=0}^{\infty} a_n x^n \ (R_1)$

(2) $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \ (R_2)$ имат един и същи радиус на сходимост

(3) $\sum_{n=0}^{\infty} n.a_n x^{n-1} \ (R_3)$

<u>Д-во:</u>

Нека $R_1, R_2, R_3$ – рад. на сх. на ст. редове (1), (2), (3).

Тр. да док. че $R_1 = R_2 = R_3$?

1) $\frac{1}{n+1} \le 1 \le n \ (\forall n \in N) \ |.|a_n x^{n+1}|$

(1) $\frac{a_n}{n+1}|x^{n+1}| \le |x|.|a_n x^n| \le |x|^2 |n.a_n x^{n+1}|$

Ако $x \in (-R_3, R_3) \overset{?}{\Rightarrow} x \in (-R_1, R_1)$

$\to \sum_{n=0}^{\infty} |n.a_n x^{n-1}|$ е сх $\Rightarrow \sum |x|^2 |n.a_n x^{n-1}| \Rightarrow$ (пр. за ср.)

сх. $\sum_{n=0}^{\infty} |x||a_n x^n| \Rightarrow$ сх. $\sum_{n=0} |a_n x^n| \Rightarrow x \in (-R_1, R_1) \Rightarrow$

$(-R_3, R_3) \subseteq (-R_1, R_1) \Rightarrow R_1 \le R_1 \ (*)$

Ако $x \in (-R_1, R_1) \Rightarrow \sum_{n=0}^{\infty} |a_n x^n|$ е сх $\Rightarrow$ сх. $\sum_{n=0}^{\infty} |x||a_n x^n| \Rightarrow$ (п-п за ср.

сх. $\sum_{n=0} \left|\frac{a_n}{n+1} \cdot x^{n+1}\right| \Rightarrow x \in (-R_2, R_2) \Rightarrow (-R_1, R_1) \subseteq (-R_2, R_2)$

$\Rightarrow R_1 \leq R_2$ (**)

От (*) и (**) $\Rightarrow R_3 \leq R_1 < R_2$ (I)

2) $R_2 \leq R_3$ (щуе док.)

Нека $x \in (-R_2, R_2) \Rightarrow 0 < |x| < t < R_2$

(2) $|n \cdot a_n x^{n-1}| = \left|\frac{a_n}{n+1} \cdot t^{n+1} \cdot \frac{x^{n+1}}{t^{n+1}} \cdot \frac{n(n+1)}{x^2}\right| =$

$= \left|\frac{a_n}{n+1} \cdot t^{n+1}\right| \cdot \left|\frac{x}{t}\right|^{n+1} \cdot \frac{n(n+1)}{x^2} \leq (\heartsuit)$

Т.к. $x \in (-R_2, R_2) \Rightarrow \sum_{n=0} \left|\frac{a_n}{n+1} \cdot x^{n+1}\right|$ е сх.

$\Rightarrow \exists M > 0 : \forall n \in \mathbb{N} \cup \{0\} \Rightarrow \left|\frac{a_n}{n+1} \cdot x^{n+1}\right| \leq M$ ($\Delta$)

$\heartsuit, \Delta \Rightarrow \leq \underbrace{\frac{M}{x^2}}_{=M_0} \cdot n(n+1) q^{n+1}$, където $q = \left|\frac{x}{t}\right| = \frac{|x|}{t} < 1$

Разглеждаме реда $\sum_{n=0}^{\infty} M_0 \cdot n(n+1) \cdot q^{n+1} = M_0 \sum_{n=0}^{\infty} n(n+1) \cdot q^{n+1}$

// $\frac{b_{n+1}}{b_n} = \frac{(n+1)(n+2) q^{n+2}}{n(n+1) \cdot q^{n+1}} = \left(1 + \frac{2}{n}\right) q \xrightarrow{n\to\infty} q < 1$ сх. сгл. кр. на Даламбер

От (2) сгл. пр. за сравн. $\Rightarrow \sum_{n=1}^{\infty} |n \cdot a_n x^{n-1}|$ е сх $\Rightarrow x \in (-R_3, R_3)$

$\Rightarrow (-R_2, R_2) \subset (-R_3, R_3)$

$\Rightarrow R_2 \leq R_3$ (II)

$\xRightarrow{\text{I и II}} R_1 = R_2 = R_3$ т.к. $R_2 \leq R_3 \leq R_1 \leq R_2$

<u>Т</u> Ако ст. ред $\sum_{n=0}^{\infty} a_n x^n$ $\left(\sum_{n=0}^{\infty} a_n (x-x_0)^n\right)$ има радиус на сх. $R > 0$ и

$f(x) = \sum_{n=0} a_n x^n$ $\left(f(x) = \sum_{n=0} a_n (x-x_0)^n\right)$, то:

1) $f(x)$ е диференцируема в/у $(-R, R)$ и

$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ $\left(f'(x) = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}\right)$, $\forall x \in (-R, R)$

2) $f(x)$ е интегруема в/у $(-R, R)$

$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ $\left(\int_{x_0}^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-x_0)^{(n+1)}\right)$,

$\forall x \in (-R, R)$

<u>Деф</u> Нека $f(x)$ е деф в $(x_0 - h, x_0 + h)$ и $\forall n \in \mathbb{N}$: $\exists f^{(n)}(x_0)$

степенния ред от вида $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$ се нарича ред на Тейлър

(Ако $x_0 = 0 \Rightarrow$ ред на Маклорен)

Ако $\hat{f}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$ в/у $(x_0 - R, x_0 + R)$

$\hat{f}(x) \stackrel{?}{=} f(x)$ в/у $(x_0 - R, x_0 + R) \cap (x_0 - h, x_0 + h)$

<u>Примери</u> 1) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\forall n \in \mathbb{N}$: $f^{(n)}(0) = 0 \Rightarrow$ ст. ред $\hat{f}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$ в/у $\mathbb{R}$

$\hat{f}(x) \equiv 0 \not\equiv f(x)$

$f(x) \to \sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!}(x-x_0)^n \Rightarrow S_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

$\underset{\Downarrow}{r_n(x)} f(x) - S_n(x)$

остатъчен член, който можем да представим чрез ф-лата на Лагр.

I] $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)$ в/у $(x_0-R, x_0+R) \cap (x_0-h, x_0+h) \overset{\Delta}{\Leftrightarrow}$

$\lim_{n\to\infty} r_n(x) = 0 \quad \forall x \in \Delta$

II] Нека $f(x)$ е деф. в/у $(x_0-h, x_0+h)$ и $\forall n \in \mathbb{N}, \exists f^{(n)}$ в/у $(x_0-h, x_0+h)$

Ако $\{f(x), f^{n}(x)\}_{n=1}^{\infty}$ са огр. съвкупност, то $\exists M>0$:

$$\left.\begin{array}{l} |f(x)| \le M \\ \text{и } |f^{(n)}(x)| \le M \end{array}\right\} \forall n \in \mathbb{N}, \forall x \in (x_0-h, x_0+h) \Rightarrow$$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ в/у $(x_0-h, x_0+h)$

<u>д-во:</u>

$r_n = f(x) - S_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^n$, където $c$ е т. м/у $x$ и $x_0$

$|r_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| |(x-x_0)^{n+1}| \le M \cdot \frac{h^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$

$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ е сх. } \forall x \in \mathbb{R}, \frac{x^n}{n!} \xrightarrow[n\to\infty]{} 0, \forall x \in \mathbb{R}\right)$

$\Rightarrow \lim_{n\to\infty} r_n(x) = 0, \quad \forall x \in (x_0-h, x_0+h)$

1) $f(x) = e^x, \forall x \in \mathbb{R}: f^{(n)}(x) = (e^x)^{(n)} e^x$

$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$(-h, h)$

$f^{(n)}(x) = e^x \Rightarrow |f^{(n)}(x)| = |e^x| \le e^h \quad \forall x \in (-h, h) \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

$\forall x \in (-h, h)$

$x = 1 \Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!}$

2) $f(x) = \sin x$

$f^{(n)}(x) = (\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$

$f^{(n)}(0) = \begin{cases} 0, & n = 2k \\ (-1)^k & n = 2k+1 \end{cases}, \quad R = +\infty$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

3) $f(x) = \cos x$

$f^{n}(x) = \cos x = \cos\left(x + \frac{\pi \cdot n}{2}\right)$

$\cos x = \sum_{k=0} \frac{(-1)^n}{(2k)!} x^{2k}, \quad R = +\infty$

$\operatorname{sh} x = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

$\operatorname{ch} x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

4) $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \ R=1$


$\alpha \in R \quad (1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{n}{\alpha} x^n, \ R=1$

$\alpha \in R \to \binom{n}{\alpha} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \qquad \left(\binom{n}{0}=1\right)$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ R=1$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n, \ R=1$

$(arctg\,x)' = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \ R=1$

$arc\,\cancel{co}tg\,x = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

20. Функции на две независими променливи величини - граница, непрекъснатост.

$\underline{X} \subset R$

$f: \forall x \in \underline{X} \xrightarrow[\text{ед.}]{f} y \in R, \ y = f(x)$ } ф-я на 1 независима пром. величина

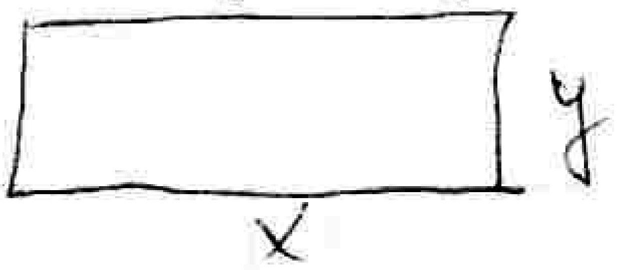
$R^2 = R \times R = \{(x,y): x, y \in R\}$

$\underline{X} \subset R^2$

$f: \underline{X} \to R$

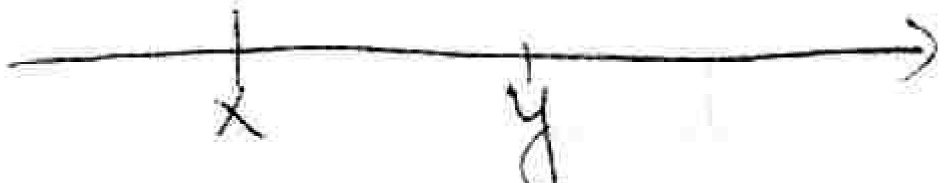
пр. ~~fe~~: $\forall (x,y) \in \underline{X} \xrightarrow[\text{ед.}]{f} z \in R$

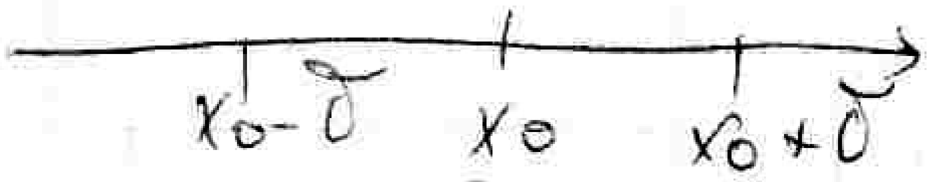
$z = f(x,y)$




$S = S(x,y) = x \cdot y \quad (x>0, \ y>0)$

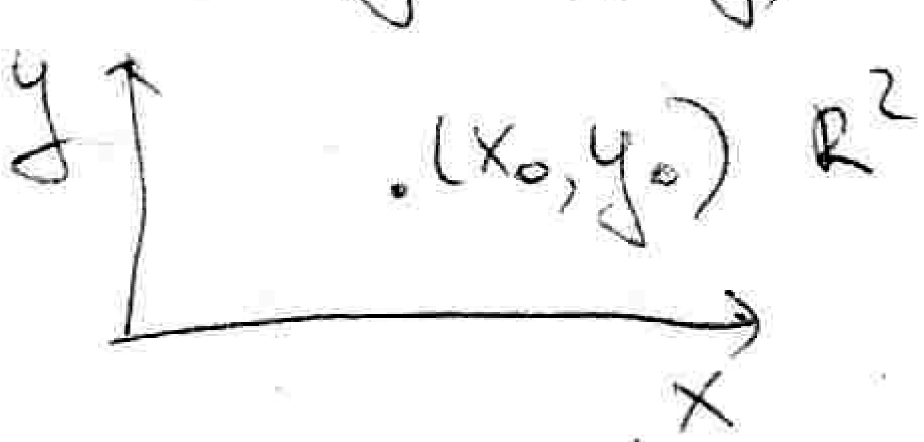
Примери: $f(x,y) = x.y$ $\quad \begin{cases} x = t \\ y = e^t \end{cases}$ $\quad \begin{cases} x = uv \\ y = ue^v \end{cases}$ $\quad z = (u,v) = f(uv, ue^v) = uv.ue^v = u^2 v e^v$




$d(x,y) = |x-y|$

$(x_0-\delta, x_0+\delta) = \{x: |x-x_0| < \delta\} = \{x: d(x,x_0) < \delta\}$




$\delta > 0$

$B_\delta(x_0,y_0) = \{(x,y): d((x,y),(x_0,y_0)) < \delta\}$

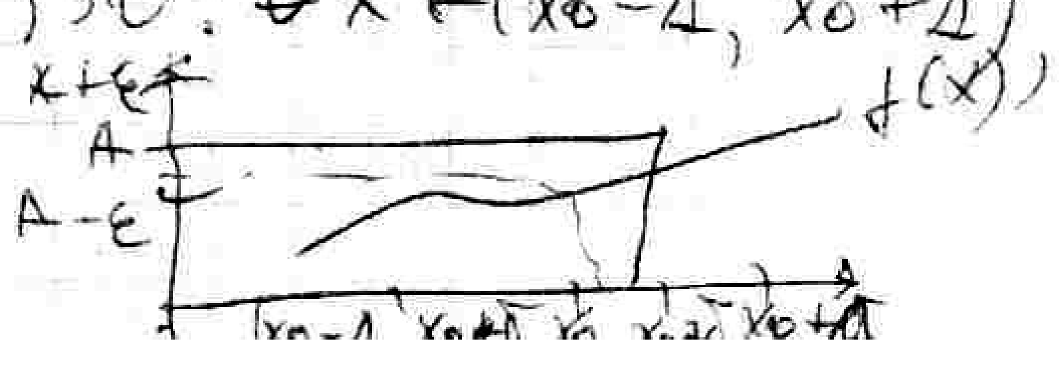
$d((x,y),(x_0,y_0)) = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

$S_\delta(x_0,y_0) = \{(x,y): d((x,y),(x_0,y_0)) = \delta\}$ - окр.

Нека ф-та $f(x)$ е деф. $(x_0-\Delta, x_0+\Delta) \setminus \{x_0\}$

Df $A = \lim_{x \to x_0} f(x)$, ако $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0: \forall x \in (x_0-\Delta, x_0+\Delta)$

$x \neq x_0, \ |x-x_0| < \delta \Rightarrow |A - f(x)| < \varepsilon$



**Деф** Нека $z = f(x,y)$ е деф $B_\Delta(x_0,y_0) \setminus \{(x_0,y_0)\}$

Казваме, че $A = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$, ако: $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$:

$\forall (x,y) \in B_\Delta(x_0,y_0)$, $(x,y) \neq (x_0,y_0)$ и $d((x,y),(x_0,y_0)) < \delta$

$\Rightarrow |A - f(x,y)| < \varepsilon$

**Свойство 1)** $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = A$ е единствена

2) **Деф** Казваме, че ф-та $f(x,y)$ е огр. в/у мн. $X$ ск, ако $\exists M > 0: \forall (x,y) \in X \Rightarrow |f(x,y)| \leq M$

$\Rightarrow$ 2) Ако $\exists \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$, то $\exists B_\delta(x_0,y_0): f(x,y)$ е огр. в/у

$B_\delta(x_0,y_0) \setminus \{(x_0,y_0)\}$

3) Ако $\exists \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = A$, $A \neq 0$, то

3.1) Ако $A > 0$, то $\exists B_\delta(x_0,y_0): \forall (x,y) \in B_\delta(x_0,y_0)$,

$(x,y) \neq (x_0,y_0) \Rightarrow f(x,y) > 0$

3.2) Ако $A < 0$, то --- $f(x,y) < 0$

4) ~~3.2)~~ Ако $f(x,y) \leq g(x,y)$ $\forall (x,y) \in B_\Delta(x_0,y_0) \setminus \{(x_0,y_0)\}$

$$\left.\begin{array}{l} \text{и } \exists \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = A \\ \exists \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = B \end{array}\right\} \text{ то } A \leq B$$

5) Ако $\exists \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ и $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)$, то $\Leftarrow\Rightarrow$

$\exists \lim\limits_{(x,y)\to(x_0,y_0)} (f \pm g)(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \pm \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)$ $\left(\text{при } \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) \neq 0\right)$ $\div$

6) Ако $f(x,y) \leq h(x,y) \leq g(x,y)$ в/у $B_\Delta(x_0,y_0) \setminus \{(x_0,y_0)\}$ и

$\exists \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ и $\exists \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)$ и те са равни на $l \Rightarrow$

$\exists \lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = l$ (т-ма за 2$^{\text{та}}$ полицаи)

* д-ва за в

**D.** Нека $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n) \ldots$ Казваме, че $\lim\limits_{n\to\infty}(x_n,y_n) = (x_0,y_0)$, ако $\forall \varepsilon > 0, \forall n > N \Rightarrow d((x_n,y_n),(x_0,y_0)) < \varepsilon$

**Т.** Б. редица от точки $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n) \to (x_0,y_0) \Leftarrow\Rightarrow$

$x_1, x_2, \ldots, x_n, \ldots \to x_0$ и $y_1, y_2, \ldots, y_n, \ldots \to y_0$

Д-во:

⇒| Нека $\lim_{n\to\infty}(x_n,y_n)=(x_0,y_0) \Rightarrow \forall \varepsilon>0, \exists N=N_\varepsilon, \forall n>N \Rightarrow$

$$d((x_n,y_n),(x_0,y_0))<\varepsilon$$

$$\varepsilon > d((x_n,y_n),(x_0,y_0)) = \sqrt{(x_n-x_0)^2+(y_n-y_0)^2} \geq \begin{cases}|x_n-x_0|\\ |y_n-y_0|\end{cases}$$

$\Rightarrow \lim_{n\to\infty} x_n = x_0, \lim_{n\to\infty} y_n=y_0$

⇐| Нека $x_1,x_2,\dots,x_n\to x_0$ и $y_1,y_2,\dots,y_n,\dots\to y_0 \Rightarrow \forall\varepsilon>0,\exists N=N_\varepsilon:$
$\forall n>N \to |x_n-x_0|<\frac{\varepsilon}{2}$ и $|y_n-y_0|<\frac{\varepsilon}{2}$

$$d((x_n,y_n),(x_0,y_0))=\sqrt{(x_n-x_0)^2+(y_n-y_0)^2} = \sqrt{|x-x_0|^2+|y_n-y_0|^2} < \sqrt{\left(\frac{\varepsilon}{2}\right)^2+\left(\frac{\varepsilon}{2}\right)^2} = \sqrt{\frac{\varepsilon^2}{2}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \lim_{n\to\infty}(x_n,y_n)=(x_0,y_0)$

Пример: $\lim_{n\to\infty}\left(\frac{1}{n}, 1-\frac{1}{n}\right) = \left(\lim_{n\to\infty}\frac{1}{n}, \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\right) = (0,1)$

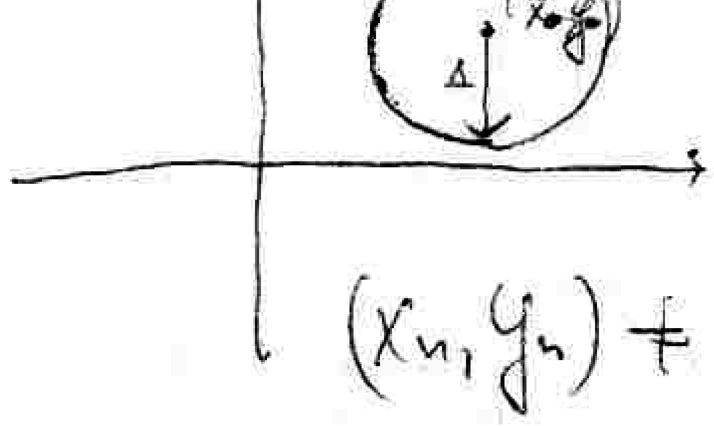
• $\lambda(x,y)=(\lambda x,\lambda y)$ // точка с число

• $(x,y)+(z,t)=(x+z,y+t)$ // сума на точки

Def(~~Коши~~ ~~Хайне~~)| Нека $f(x)$ деф. в/у $(x_0-\Delta, x_0+\Delta)\setminus\{x_0\}$. Казваме, че
редица от точки
$A = \lim_{x\to x_0} f(x)$, ако $\forall\ x_1,x_2,\dots,x_n \to x_0: x_n\neq x_0,\ x_n\in(x_0-\Delta,x_0+\Delta) \Rightarrow$

$$f(x_1),f(x_2),\dots,f(x_n)\to A$$

Def(Коши ~~Хайне~~)| Нека $z=f(x,y)$ е деф. в/у $B_\Delta(x_0,y_0)\setminus\{(x_0,y_0)\}$.



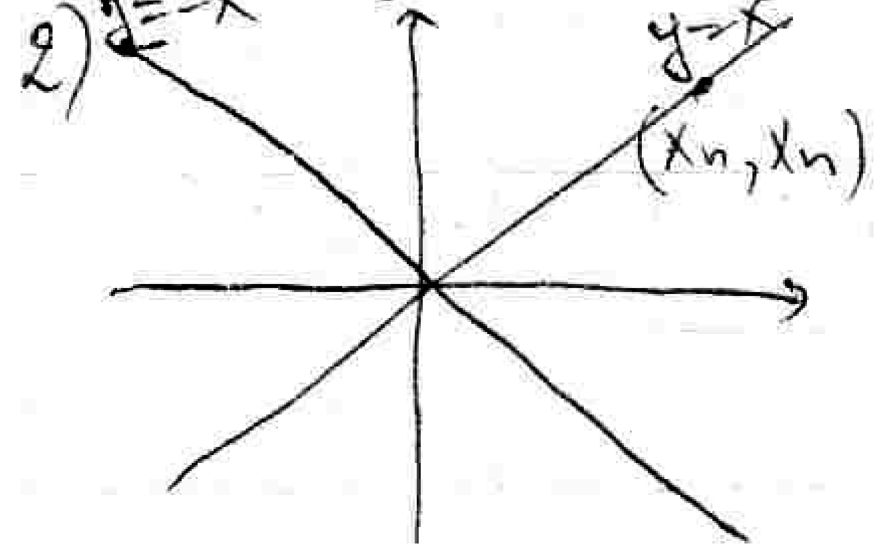

Казваме, че $A=\lim_{(x,y)\to(x_0,y_0)} f(x_0,y_0)$, ~~$(x,y)\to(x$~~ ако
$\forall\ (x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\to(x_0,y_0)$
$(x_n,y_n)\neq(x_0,y_0),\ (x_n,y_n)\in B_\delta(x_0,y_0) \Rightarrow f(x_1,y_1),f(x_2,y_2),\dots,f(x_n,y_n)\to A$

Коши ⇔ Хайне

Пример| 1) $f(x,y)=\sqrt[3]{x^2+y^2}$
$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\sqrt[3]{x^2+y^2}$
$\forall (x_n,y_n)\xrightarrow[n\to\infty]{}(0,0) \Leftrightarrow$ // произволна, фиксирана

$\Leftrightarrow \begin{cases}x_n\to 0\\ y_n\to 0\end{cases} \Rightarrow \begin{cases}x_n^2\to 0\\ y_n^2\to 0\end{cases} \Rightarrow x_n^2+y_n^2\to 0+0=0 \xrightarrow{\text{непр.}}$

$\sqrt[3]{x_n^2+y_n^2}\to\sqrt[3]{0} \Rightarrow \lim_{(x,y)\to(0,0)}\sqrt[3]{x^2+y^2}=0$

2)



$f(x,y)=\frac{2xy}{x^2+y^2}$, $D(f)=\mathbb{R}^2\setminus\{(0,0)\}$

i) $\{(x_n,x_n)\}_{n=1}^{\infty}$, $x_n\neq 0$

$(x_n,x_n)\xrightarrow[n\to\infty]{}(0,0)$

$\forall n \in \mathbb{N}\ f(x_n, x_n) = \frac{2x_n^2}{x_n^2+x_n^2} = \frac{2x_n^2}{2x_n^2} \xrightarrow[n\to\infty]{} 1$

ii) $(x_n, -x_n)$, $x_n \neq 0$, $(x_n,-x_n) \xrightarrow[n\to\infty]{} (0,0)$

$\forall n \in \mathbb{N}: f(x_n,-x_n) = \frac{-2x_n^2}{x_n^2+(-x_n^2)} = -\frac{2x_n^2}{2x_n^2} = -1 \xrightarrow[n\to\infty]{} -1$

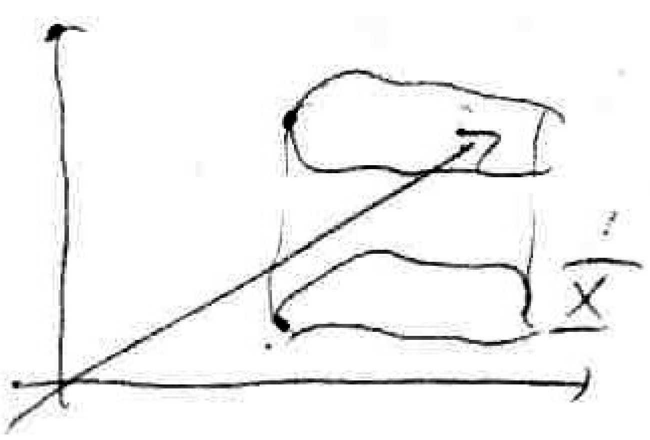
$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ // i и ii не клонят към 1 число

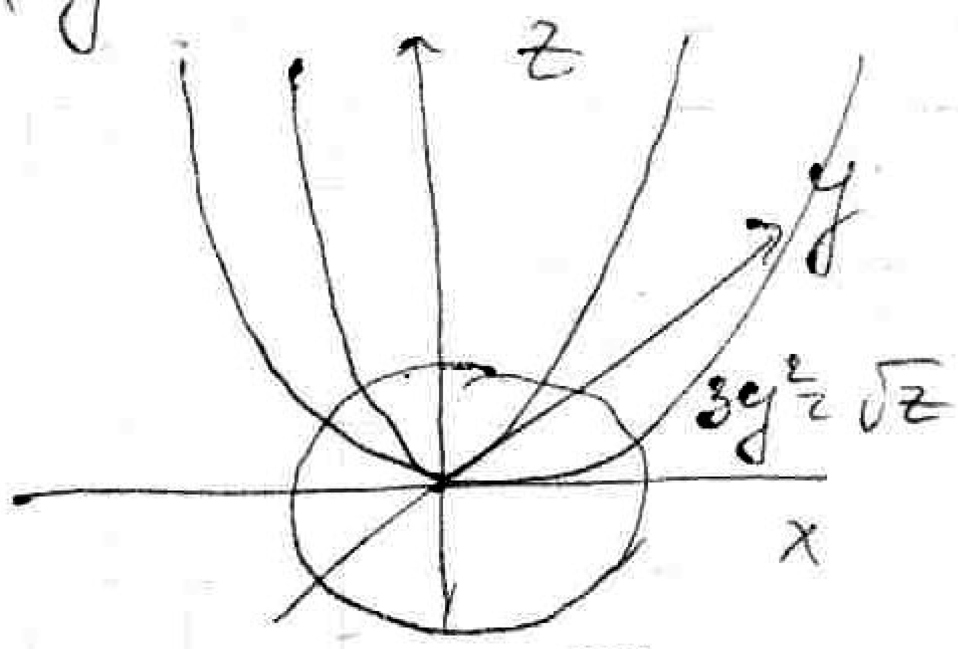
Def| $z=f(x,y)$ деф. в/у $X \subset \mathbb{R}^2$

$\Gamma_f = \{((x,y), f(x,y)) : (x,y)\in X\}$

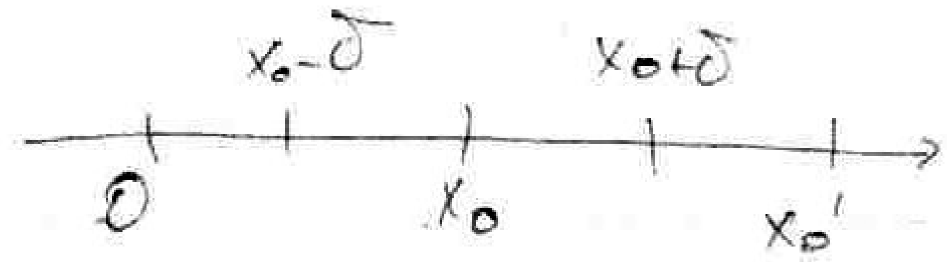
↳ графика на ф-я

Пример:| $z = f(x,y) = x^2+y^2$



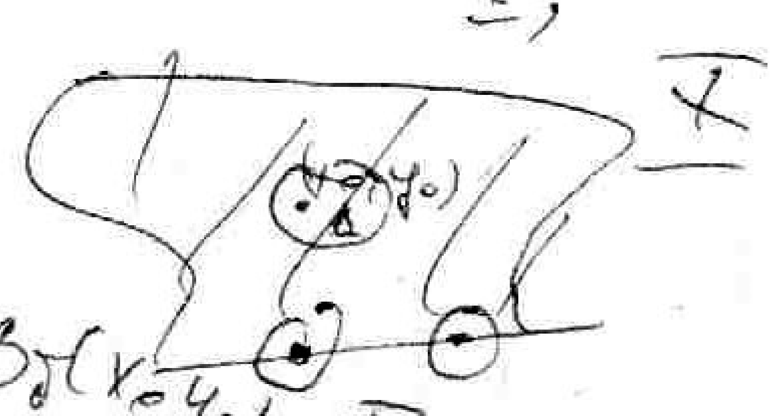




$x^2+y^2=r^2$ / окр. с рад. r




Def| Казваме, че $x_0$ – т. на сг. на $X$, ако $\forall \delta>0: X \cap [(x_0-\delta, x_0+\delta)\setminus\{x_0\}] \neq \emptyset$

Def| $X\subset\mathbb{R}^2$, $(x_0,y_0)\in\mathbb{R}^2$. Казваме, че $(x_0,y_0)$ е т. на сг. на $X$, ако $\forall \delta>0: X\cap B_\delta(x_0,y_0)\setminus\{(x_0,y_0)\}\neq\emptyset$




Def| $X\subset\mathbb{R}^2$, $(x_0,y_0)\in X$. Казваме, че $(x_0,y_0)$ е

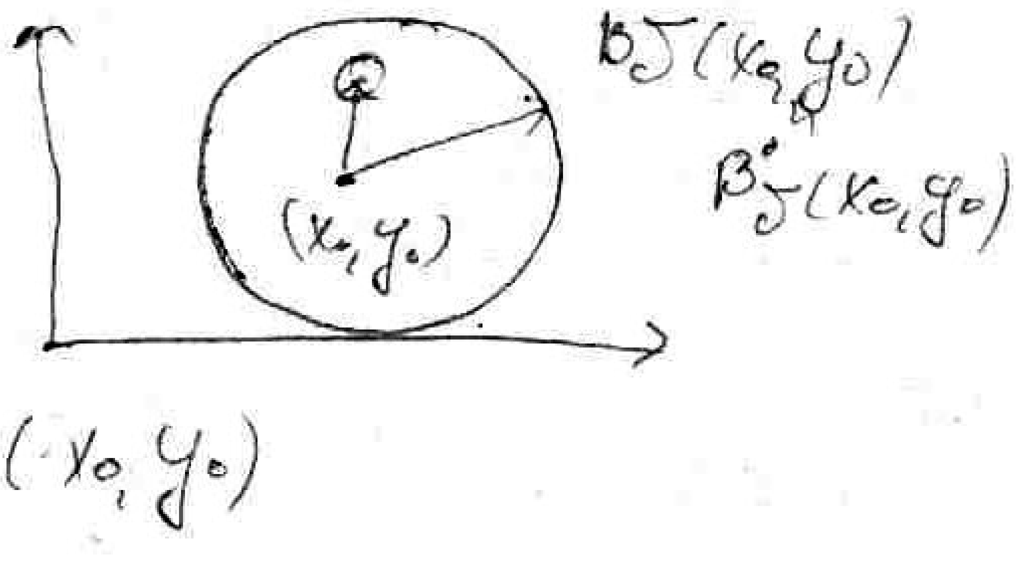
вътрешна т. на мн. $X$, ако $\exists B_\delta(x_0,y_0): B_\delta(x_0,y_0)\subset X$

$X$ – множ. от $\forall$ вътр. т. $X^\circ = \text{int}\, X$ (обозначение)

$X \cup \{$т. на сг.$\} = \overline{X}$: затворена обвивка на $X$

0 ( ] 1 → $(0,1]\cup\{$т. на сг.$\} = (0,1]\cup[0,1] = [0,1] \Rightarrow$

$[0,1] = \overline{(0,1]}$




Def| Ако $X = X^\circ \Rightarrow X$ – отв. мн.

Def| Ако $X = \overline{X} \Rightarrow X$ – затв. мн.

Def| $\partial X = \overline{X}\setminus X^\circ$ – контур на $X$ $\Rightarrow$ $\overline{B_\delta}\setminus B_\delta^\circ = S_\delta(x_0,y_0)$

Def| $B_\delta(x_0,y_0) = \{(x,y): d((x,y),(x_0,y_0))<\delta\}$

$\overline{B_\delta}(x_0,y_0) = \{(x,y): d((x,y),(x_0,y_0))\le\delta\}$

$\overline{B_\delta}(x_0,y_0)\setminus B^\circ_\delta(x_0,y_0) = S_\delta(x_0,y_0) = \{(x,y): d((x,y),(x_0,y_0))=\delta\}$

0 ( ) 1 → отв. мн.

0 [ ] 1 → отв. м-во + т. на сг. $[0,1]\setminus(0,1) = \{0,1\}$ → контур

т. от контура – гранични точки

Деф| Нека $f(x,y)$ е деф. в/у $X \subset \mathbb{R}^2$, $(x_0,y_0)$ – т. на сг. на $X$.

Казваме, че $A = \lim\limits_{\substack{(x,y)\to(x_0,y_0)\\ (x,y)\in X}} f(x,y)$, ако $\forall\ \varepsilon>0$, $\exists\ \delta=\delta(\varepsilon)>0 : \forall (x,y)\in X$

$(x,y)\neq(x_0,y_0)$, $d((x,y),(x_0,y_0))<\delta \Rightarrow |f(x,y)-A|<\varepsilon$

// ако $(x,y)$ – вътр. т. – няма разлика с предната деф

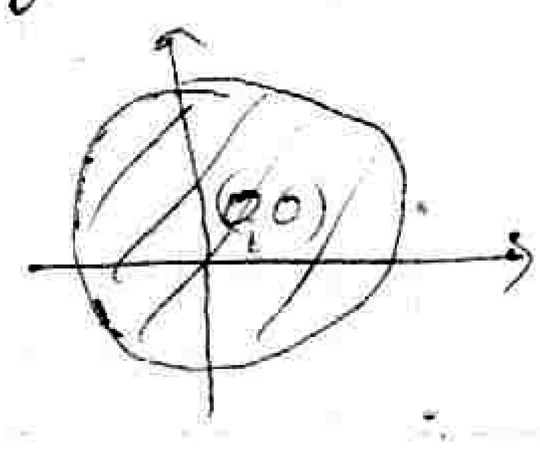
// ако е гранична !!! ~ аналог на лява и дясна гр.

(Хайне) $\forall\ (x_1,y_1),(x_2,y_2),\dots,(x_n,y_n) \dots \to (x_0,y_0)$, $(x_n,y_n)\in X$, $\neq(x_0,y_0)$

$\Rightarrow f(x_1,y_1), f(x_2,y_2) \dots, f(x_n,y_n) \to A$

Пример:| $f(x,y) = \dfrac{2xy}{x^2+y^2}$, $X = \mathbb{R}^2 \setminus \{(0,0)\}$

$\nexists \lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)\in X}} f(x,y)$.



$X' = \{(x,x),\ x\neq 0\}$ // взимаме същата ф-я, др. множ.

$\lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)\in X'}} \dfrac{2xy}{x^2+y^2} = 1$

## Непрекъснатост на ф-я

D1 (за ф-я на 1 пром.) Нека $f(x)$ е деф. в/у $(x_0-\Delta, x_0+\Delta)$
Казваме, че $f(x)$ е непр. в т. $x_0$, ако $\lim\limits_{x\to x_0} f(x) = f(x_0)$

D (за 2 пром.)| Нека $z=f(x,y)$ е деф. в/у $B\amalg(x_0,y_0)$
Казваме, че $f(x,y)$ е непр. в т. $(x_0,y_0)$, ако $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

D.1 Нека $z=f(x,y)$ е деф. в/у $X\subset\mathbb{R}^2$ и $(x_0,y_0)\in X$.

Казваме, че $f(x,y)$ е непр. в т. $(x_0,y_0)$ по мн. $X$, ако

$\lim\limits_{\substack{(x,y)\to(x_0,y_0)\\ (x,y)\in X}} f(x,y) = f(x_0,y_0)$ // непр по дадено м-во


Пример: $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

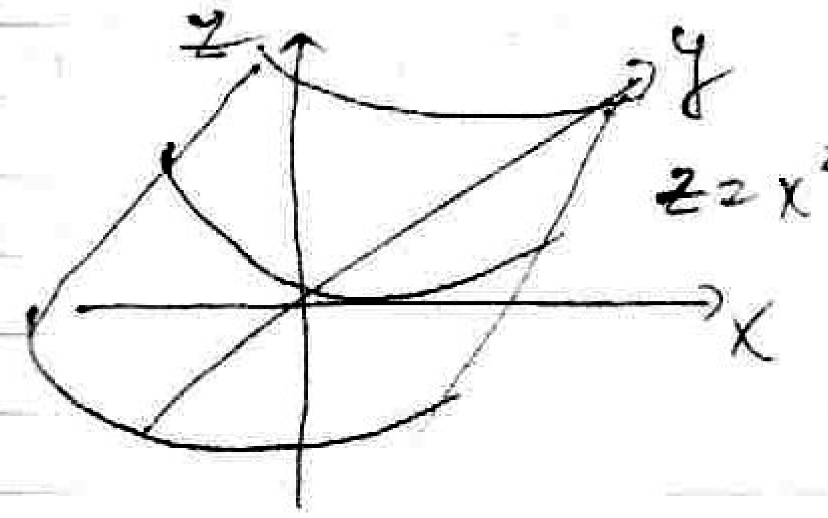
$f(x,y) \neq$ непр в т. $(0,0)$ з-то $\nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)\in X}} = \lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)=(x,0)}} \dfrac{2x\cdot 0}{x^2+0^2} = \lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)=(x,0)}} 0 = 0 = f(0,0) \Rightarrow f$ е непр. в $(0,0)$

НО! в това м-во $X$ → в общия смисъл е прекъсната

$y = g(x) = x^2$ // $\mathbb{R}: (-\infty,+\infty) \to$ непр

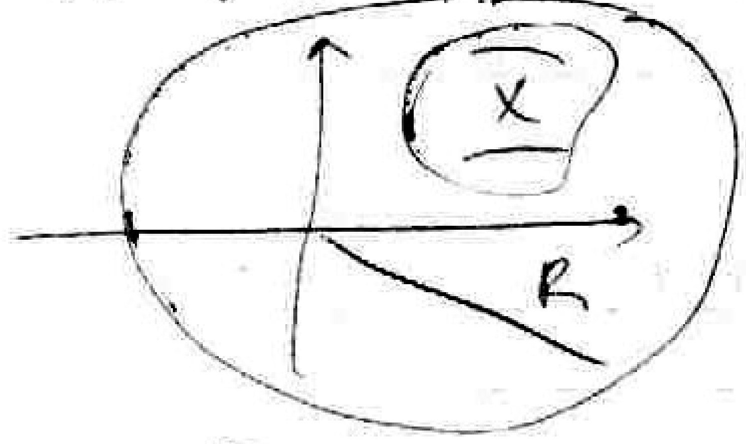
$z = f(x,y) = x^2$ // разгл. я като ф-я на 2 пром. $\Rightarrow$ пак е непр. ф-я

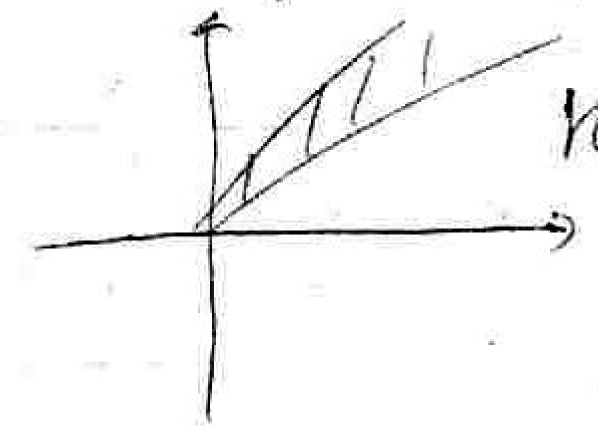



(като улук)

Деф| Казваме, че $f(x,y)$ е непр. в/у $\overline{X} \subset \mathbb{R}^2$, ако $f(x,y)$ е непр. във $\forall$ т. $(x,y) \in \overline{X}$

Деф| Мн. $\overline{X} \subset \mathbb{R}^2$ се нарича огр., ако $\exists\, B_R(0,0): X \subset B_R(0,0)$
// $B_R(0,0)$ - кръг с ц-р $(0,0)$ и рад. $R$)




Деф| Мн. $\overline{X} \subset \mathbb{R}^2$ се нарича компактно, ако $\overline{X}$ е огр. и затв.

Т1| Ако ф-та $f(x,y)$ е непр. в/у комп. м-во $\overline{X} \subset \mathbb{R}^2 \Rightarrow$
$f(x,y)$ е огр. в/у $\overline{X}$, т.е. $\exists\, M>0,\ \forall (x,y) \in \overline{X} \Rightarrow |f(x,y)| \le M$

Т2| Ако ф-та $f(x,y)$ е непр. в/у комп. м-во $\overline{X} \subset \mathbb{R}^2 \Rightarrow$
$\exists\,(x_0,y_0), (x_1,y_1) \in \overline{X}: \forall (x,y) \in \overline{X} \Rightarrow f(x_0,y_0) \le f(x,y) \le f(x_1,y_1)$
т.е. $f(x_0,y_0) = \min\limits_{(x,y)\in\overline{X}} f(x,y)$, $f(x_1,y_1) = \max\limits_{(x,y)\in\overline{X}} f(x,y)$ // Док. като в Анализ 1

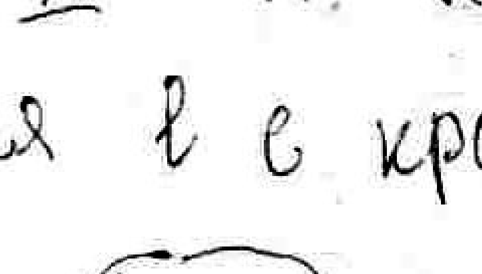
Деф| Множ. $l = \{(x,y): x = \underbrace{x(t), y = y(t)}_{\text{непр. ф-ции}}, t \in [\alpha,\beta]\}$ се нарича крива линия в $\mathbb{R}^2$

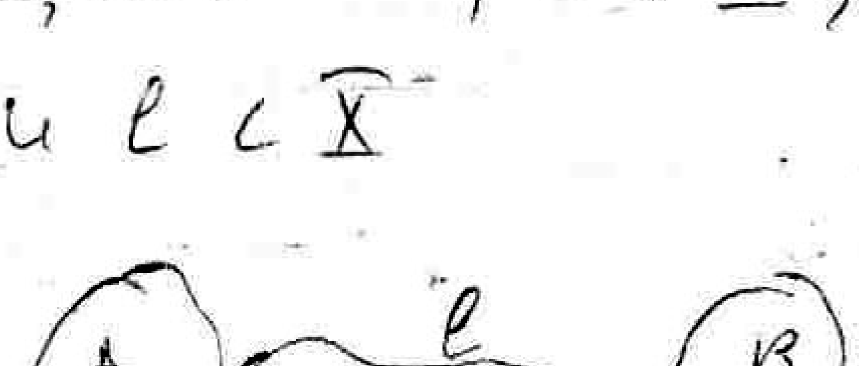
$l: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, t \in [\alpha,\beta]$. Т. $A(x(\alpha), y(\alpha))$ и $B(x(\beta), y(\beta))$ - краища на $l$

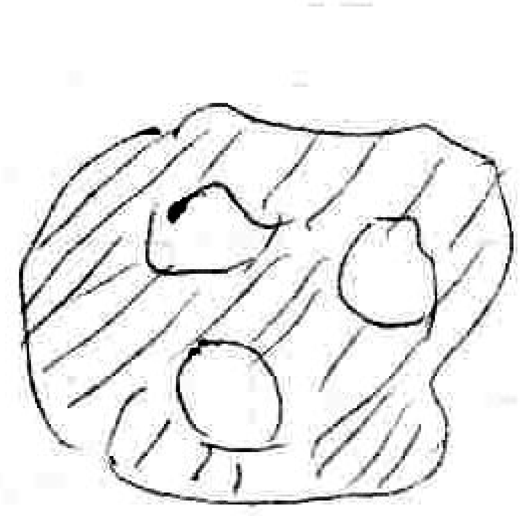
Пример: $\underset{\text{полуокр.}}{l}: \begin{cases} x = t \\ y = \sqrt{1-t^2} \end{cases}, t \in [-1,1]$


Деф| Мн. $\overline{X} \subset \mathbb{R}^2$ наричаме свързано, ако $\forall$ т. $A, B \in \overline{X}$, $\exists$ крива линия $l$ с краища т. $A$ и $B$ и $l \subset \overline{X}$

Пример|



I] Ако ф-та $f(x,y)$ е непр. в/у свързаното м-во $\overline{X} \subset R^2$ и

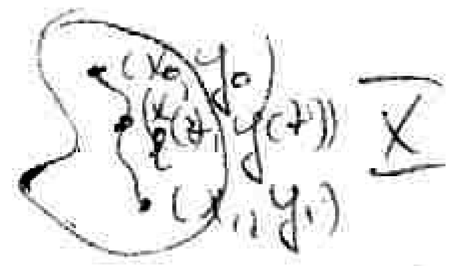
$\exists\ (x_0,y_0)(x_1,y_1) \in \overline{X},\ f(x_0,y_0)\cdot f(x_1,y_1) < 0 \Rightarrow$:

$\exists\ (c,d) \in \overline{X} : f(c,d)=0$

Д-во:

т.к. $\overline{X}$ е свързано м-во $\Rightarrow \exists$ криви линии $l$ (непр.)

$l: \begin{cases} x=x(t) \\ y=y(t) \end{cases},\ t\in[\alpha,\beta]$ и $\begin{matrix}(x(\alpha),y(\alpha))=(x_0,y_0) \\ (x(\beta);y(\beta))=(x_1,y_1)\end{matrix}$ и $l \subset \overline{X}$




Образуваме ф-та $h(t)=f(x(t);y(t))$, $t\in[\alpha,\beta]$ (непр. в/у $[\alpha,\beta]$)

и $h(\alpha).h(\beta)=f(x(\alpha),y(\alpha)).f(x(\beta),y(\beta))=$

$= f(x_0,y_0).f(x_1,y_1)<0 \xRightarrow{\text{т. от Ан. I}} \exists\ t_0\in(\alpha,\beta): h(t_0)=0$, но $h(t_0)=f(x(t_0),y(t_0))=0$

т.е. в т. $(c,d)=(x(t_0),y(t_0))$ ф-та $f(x,y)$ се анулира в $(c,d)$

Деф] Нека $f(x,y)$ е деф. в/у $\overline{X}\subset R^2$. Казваме, че $f(x,y)$ е равном. непр. в/у $\overline{X}$, ако $\forall\ \varepsilon>0,\ \exists\ \delta=\delta(\varepsilon)>0$:

$\forall (x',y'),(x'',y'')\in\overline{X}$ : $d((x',y'),(x'',y''))<\delta \Rightarrow$

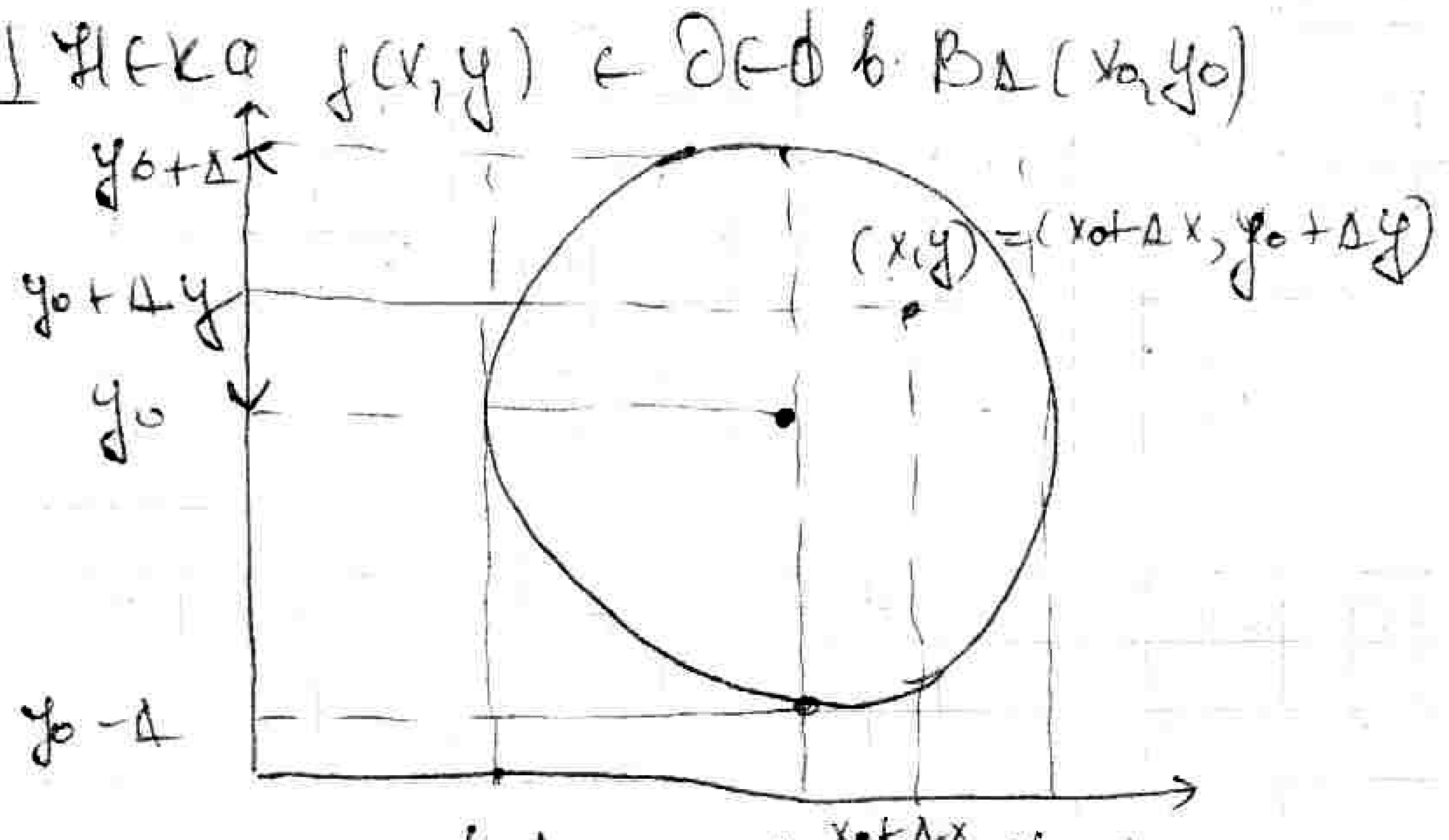
$|f(x',y')-f(x'',y'')|<\varepsilon$

Т] Ако ф-та $f(x,y)$ е непр. в/у комп. м-во $\overline{X}\subset R^2 \Rightarrow$

$f(x,y)$ е равном. непр. в/у $\overline{X}$ // Док. от Ан. I

21. Частни производни. Диференцируемост, пълен диференциал. Достат. усл. за диференцируемост. Частни пр. от по-висок ред, равенство на смесените производни.

D] Нека $f(x,y)$ е деф. в $B_\Delta(x_0,y_0)$




$\varphi(x)=f(x,y_0)$

$x\in(x_0-\Delta,\ x_0+\Delta)$

Ако $\exists\ \varphi'(x_0)$, то тази производна е първа частна производна на $f(x,y)$ в т. $(x_0,y_0)$ по пром. $x$.

$\varphi'(x_0)=\dfrac{\partial f(x_0,y_0)}{\partial x}=f'_x(x_0,y_0)=$

$f_x(x_0,y_0)$

$\varphi'(x_0)=\lim\limits_{x\to x_0}\dfrac{\varphi(x)-\varphi(x_0)}{x-x_0}=$

$=\lim\limits_{x\to x_0}\dfrac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$.

Аналог, ако $\psi(y)=f(x_0,y)$, $y\in(y_0-\Delta,\ y_0+\Delta)$, то $\psi'(y_0)$ се нарича първа ч. пр. на $f(x,y)$ в т. $(x_0,y_0)$ по пром. $y$.

$\psi'(y_0) = \frac{\partial f(x_0, y_0)}{\partial y} = f'_y(x_0, y_0) = f_y(x_0, y_0)$ – озн.

$\frac{\partial f(x,y)}{\partial x}$ – оператор, не частно!

Пример: $f(x,y) = xy^2 e^{x-y}$

$\frac{\partial f}{\partial x} = (xy^2 e^{x-y})'_x = y^2 (x e^{x-y})'_x = y^2 (e^{x-y} + x . e^{x-y} . 1) =$ ↳ пр. спрямо x

$= (1+x) y^2 e^{x-y}$

$\frac{\partial f}{\partial y} = (xy^3 e^{x-y})'_y = x (y^2 e^{x-y})'_y = x (2y e^{x-y} + y^2 . e^{x-y} (-1)) =$

$= xy(2-y) e^{x-y}$

Def| $y = f(x)$ деф. в $(x_0 - \Delta, x_0 + \Delta)$. Ф-та $f(x)$ е диф. в т. $x_0$, ако $\exists A \in R$: $\Delta f = f(x) - f(x_0) = \underline{A(x - x_0)} + \alpha(x - x_0)$, където

$\lim_{x \to x_0} \alpha(x - x_0) = 0$,

$df(x_0) = A(x - x_0)$ – диференциал

$df(x_0) = f'(x_0) dx$, $f'(x_0) = \frac{df(x_0)}{dx}$

T| Ако $f(x)$ е диф. в т. $x_0$, то $f(x)$ е непр.

T| $f(x)$ е диф. в т. $x_0 \Leftrightarrow \exists f(x_0)$. При това $A = f'(x_0)$

Def| Нека $f(x,y)$ е деф. в $\Delta(x_0, y_0)$, ако $\exists A, B \in R$:

$(*)\ \Delta f = f(x,y) - f(x_0, y_0) = A(x - x_0) + B(y - y_0) + \alpha_1(x - x_0, y - y_0) . (x - x_0) +$
$+ \alpha_2(x - x_0, y - y_0)$, където $\lim_{(x,y) \to (x_0, y_0)} \alpha_i (x - x_0, y - y_0) = 0$, $i = 1, 2$

Def| $df(x_0, y_0) = A(x - x_0) + B(y - y_0)$ – диференциал

T1| Ако $f(x,y)$ е диф. в т. $(x_0, y_0) \Rightarrow f(x,y)$ е непр. в т. $(x_0, y_0)$ =

Д-во:

$f(x,y)$ е диф. в т. $(x_0, y_0) \Rightarrow$ е в сила тв. $(*) \Rightarrow$.

$\lim_{(x,y) \to (x_0, y_0)} [f(x,y) - f(x_0, y_0)] = \lim_{(x,y) \to (x_0, y_0)} [A(x - x_0) + B(y - y_0) + \alpha_1(x - x_0, y - y_0) . (x - x_0) +$
$\alpha_2(x - x_0, y - y_0)(y - y_0)] =$

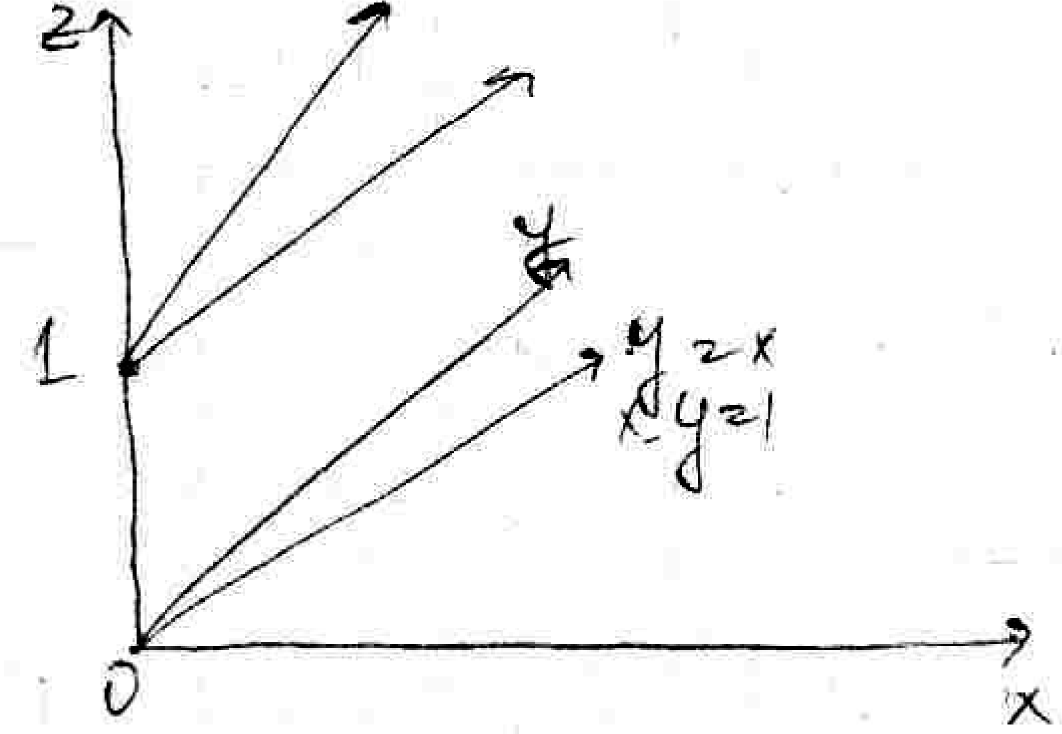
$= A . \lim_{(x,y) \to (x_0, y_0)} (x - x_0) + B \lim_{(x,y) \to (x_0, y_0)} (y - y_0) + \lim_{(x,y) \to (x_0, y_0)} \alpha_1 (x - x_0, y - y_0) \lim_{(x,y) \to (x_0, y_0)} (x - x_0) + \lim_{(x,y) \to (x_0, y_0)} \alpha_2 (x - x_0, y - y_0) . \lim_{(x,y) \to (x_0, y_0)} (y - y_0)$

$= A . B + B . 0 + 0 . 0 + 0 . 0 = \underline{0}$

$\Rightarrow \lim_{(x,y) \to (x_0, y_0)} f(x,y) = f(x_0, y_0) \overset{\text{деф.}}{\Rightarrow} f(x,y)$ е непр. в т. $(x_0, y_0)$

T2| Ако $f(x,y)$ е диф. в т. $(x_0, y_0)$ и $d(x_0, y_0) = A(x - x_0) + B(y - y_0)$ – диференциала на $f(x,y)$ в т. $(x_0, y_0) \overset{*}{\Longrightarrow} \exists \frac{\partial f(x_0, y_0)}{\partial x}, \frac{\partial f(x_0, y_0)}{\partial y}$, при това

$A = \frac{\partial f(x_0,y_0)}{\partial x}$, $B = \frac{\partial f(x_0,y_0)}{\partial y}$



<u>Пример</u>: $f(x,y) = \begin{cases} 0, & xy = 0 \\ 1, & xy \neq 0 \end{cases}$ $(\parallel Oxy)$

$\frac{\partial f(0,0)}{\partial x} = 0$, $\frac{\partial f(0,0)}{\partial y} = 0$

$f(x,y)$ е прек. в т. $(0,0)$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} f(x,x) = 1 \neq f(0,0) = 0$ => ф-та действ. е прек. в т. $(0,0)$

=> $f(x,y)$ не е диф. в т. $(0,0)$

<u>Д-во:</u>

1) $f(x,y)$ е диф. в т. $(x_0,y_0)$ => $f(x,y) - f(x_0,y_0) = A(x-x_0) + B(y-y_0) + \alpha_1(x-x_0) + \alpha_2(y-y_0)$,

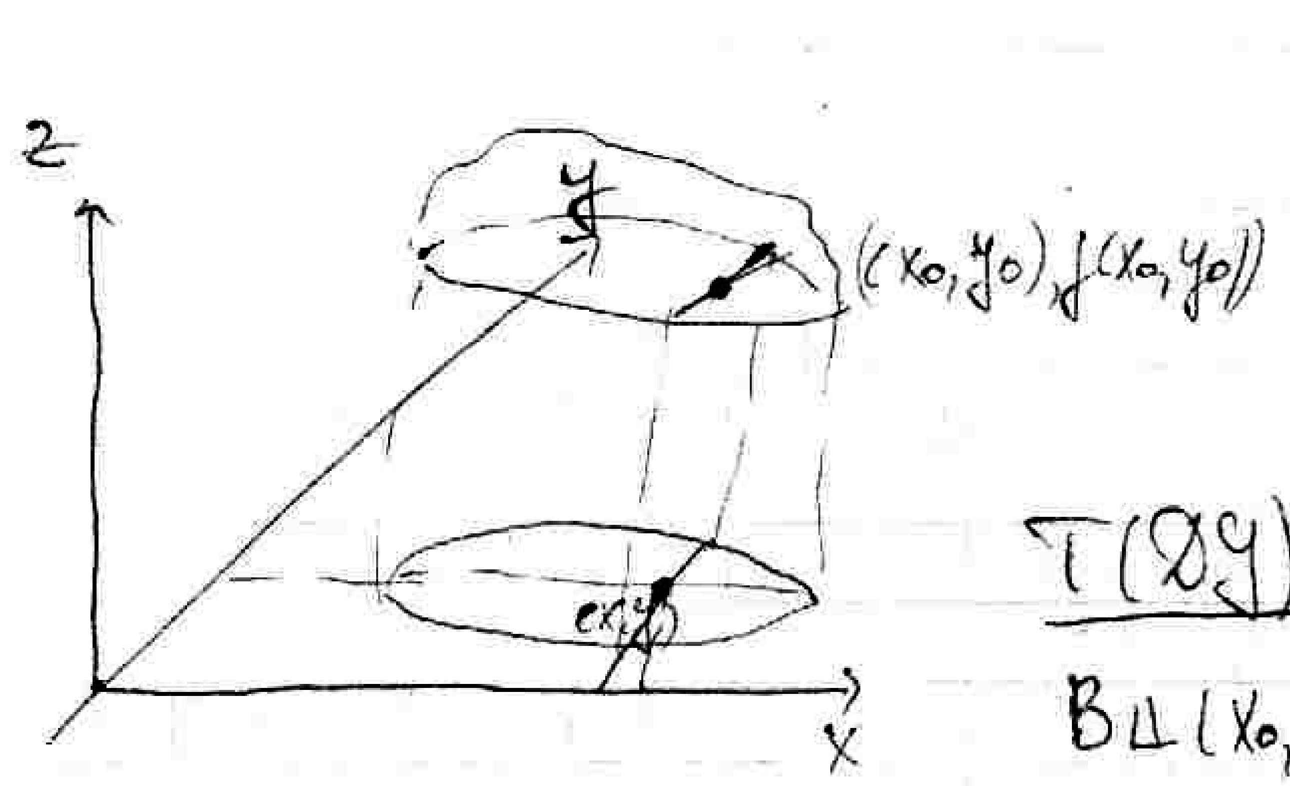
$\lim\limits_{(x,y)\to(x_0,y_0)} \alpha_i = 0$

$\lim\limits_{x\to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x-x_0} = \lim\limits_{x\to x_0} \frac{A(x-x_0) + \alpha_1(x-x_0)}{x-x_0} = \lim\limits_{x\to x_0}(A+\alpha_1) = A + \lim\limits_{x\to x_0}\alpha_1 = A+0 = A$

↳ пр. частна по x

=> $\exists \frac{\partial f(x_0,y_0)}{\partial x} = A$

2) B - аналогично



Доп. равнина: $\alpha: z = f(x_0,y_0) = \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)$

<u>Т(Д,У)</u> Нека ф-та $f(x,y)$ има $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$ в (у) $B_\Delta(x_0,y_0)$ и $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$ са непр. в т. $(x_0,y_0)$

=> $f(x,y)$ е диф. в т. $(x_0,y_0)$

<u>Д-во:</u>

$\forall (x,y) \in B_\Delta(x_0,y_0)$, $(x,y) \neq (x_0,y_0)$ и нека $\Delta x = x - x_0$, $\Delta y = y - y_0$ $\qquad \varphi(x) = f(x, y_0+\Delta y)$, $\psi(x) = f(x_0, y)$

$\Delta f = f(x,y) - f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) =$

$= [\underbrace{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)}_{\varphi(x_0+\Delta x) - \varphi(x_0)}] + [\underbrace{f(x_0, y_0+\Delta y) - f(x_0,y_0)}_{\psi(y_0+\Delta y) - \psi(y_0)}] =$

$\exists\, 0<\theta_1,\ \theta_2<1$: $f'(x)(x_0+\theta_1\Delta x, y_0+\Delta y)\cdot\Delta x + f'_y(x_0, y_0+\theta_2\Delta y)\Delta y =$

$= f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + [\underbrace{f'_x(x_0+\theta_1\Delta x, y_0+\Delta y) - f'_x(x_0 y_0)}_{\alpha_1(\Delta x,\Delta y)}]\Delta x +$

$+ [\underbrace{f'_y(x_0, y_0+\theta_2\Delta y_0) - f'_y(x_0 y_0)}_{\alpha_2(\Delta x,\Delta y)}]\Delta y =$

$= f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \alpha_1\underbrace{(\Delta x,\Delta y)}\Delta x + \alpha_2\underbrace{(\Delta x,\Delta y)}\Delta y$

тр. да док, че $\to 0$

$\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \overbrace{\alpha_1(\Delta x,\Delta y)}^{x} = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} [f'_x(x_0+\theta_1\Delta x, y_0+\Delta y) - f_x(x_0,y_0)]$ огр. $\downarrow 0$

$(\Delta x,\Delta y)\to(0,0) \Rightarrow (x_0+\theta_1\cdot\Delta x, y_0+\Delta y)\to(x_0,y_0) \Rightarrow f'_x(x_0+\theta_1\Delta x, y_0+\Delta y) \to f'_x(x_0,y_0)$

// частните пр. са непр. в $(x_0,y_0)$

$\Rightarrow x = 0$

Аналог. $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \alpha_2(\Delta x,\Delta y) = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} [f'_y(x_0, y_0+\theta_2\cdot\Delta y) - f'_y(x_0,y_0)] = 0$

$\Rightarrow f(x,y)$ е диф. в т. $(x_0,y_0)$

- $f(x,y)$ е диф. в $\overset{\circ}{B_\Delta}(x_0,y_0)$ и $\exists\ \frac{\partial f(x,y)}{\partial x}$ и $\frac{\partial f(x,y)}{\partial y}$ за $\forall (x,y)\in B_\Delta(x_0,y_0)$ и:

$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial^2 f(x,y)}{\partial x^2} = f''_{x^2}$

$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial^2 f(x,y)}{\partial y\partial x} = f''_{yx}$ – смесена частна пр. от 2ри ред

$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{\partial^2 f(x,y)}{\partial x\partial y} = f''_{xy}$ см. ч. пр. от 2ри р.

$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{\partial^2 f(x,y)}{\partial y^2} = f''_{y^2}$

<u>Пример</u>: $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$

$\frac{\partial^2 f(0,0)}{\partial x\partial y} = 1 \neq \frac{\partial^2 f(0,0)}{\partial y\partial x} = -1$ – смесените пр. са $\neq$

<u>Т (рав. на смесените пр.)</u>: Нека $f(x,y)$ е диф. и $\exists\ f'_x, f'_y, f''_{xy}, f''_{yx}$ в/у $B_\Delta(x_0,y_0)$. Ако $f''_{xy}$ и $f''_{yx}$ са непр. в т. $(x_0,y_0)$, то $f''_{xy}(x_0,y_0) = f''_{yx}(x_0,y_0)$

## (22) Локален екстр. на ф-я на 2 пром.-необх. и дост. усл.

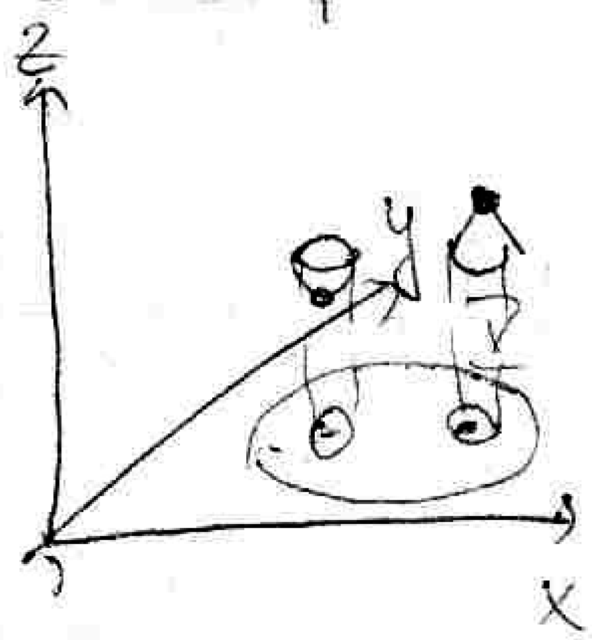
**D|** $f(x)$ е деф. в/у $X \subset R$; $x_0 \in X$

1) Казваме, че т. $x_0$ е т. на лок. max, ако $\exists (x_0-\delta, x_0+\delta) \subset X$:
$\forall x \in (x_0-\delta, x_0+\delta) \rightarrow f(x) \leq f(x_0)$

2) т. $x_0$ – т. на лок. min, ако $\exists (x_0-\delta, x_0-\delta) \subset X$:
$\forall x \in (x_0-\delta, x_0+\delta) \rightarrow f(x) \geq f(x_0)$

**D|** Нека $f(x,y)$ е деф. в/у $X \subset R^2$, $(x_0,y_0) \in X$.

Казваме, че: 1) т. $(x_0,y_0)$ е т. на лок. max, ако $\exists B_\delta(x_0,y_0) \subset X$:
$\forall (x,y) \in B_\delta(x_0,y_0) \Rightarrow f(x,y) \leq f(x_0,y_0)$



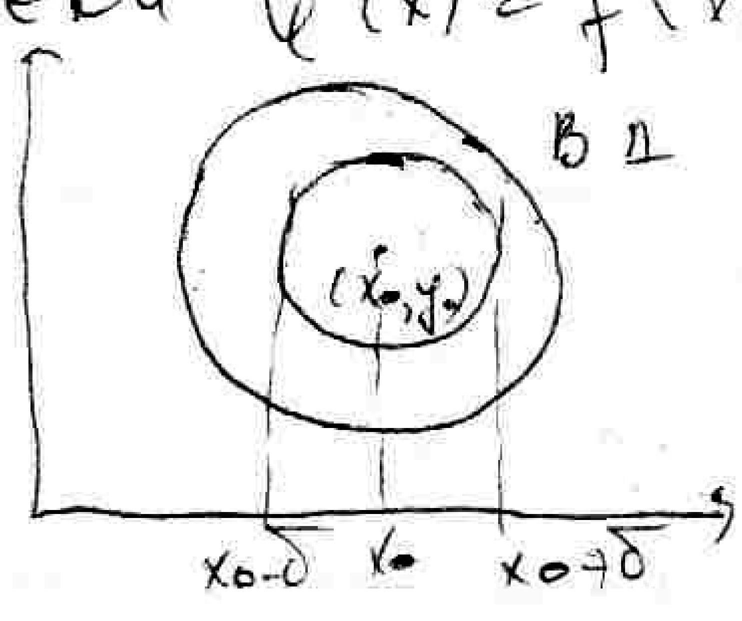
2) т. $(x_0,y_0)$ – т. на лок. min, ако $\exists B_\delta(x_0,y_0) \subset X$:
$\forall (x,y) \in B_\delta(x_0,y_0) \Rightarrow f(x,y) \geq f(x_0,y_0)$

**Т (НУ)|** Нека $f(x,y)$ е деф. в/у $B_\Delta(x_0,y_0)$ и $(x_0,y_0)$ е т. на лок. екстр. на $f(x,y)$. Ако $\exists \dfrac{\partial f(x_0,y_0)}{\partial x/y} \Rightarrow \dfrac{\partial f(x_0,y_0)}{\partial x/y} = 0$

**Д-во:**

Нека т. $(x_0,y_0)$ е т. на лок. max (за опр.) $\Rightarrow$
$\exists B_\delta(x_0,y_0)$: $\forall (x,y) \in B_\delta(x_0,y_0) \rightarrow f(x,y) \leq f(x_0,y_0)$.

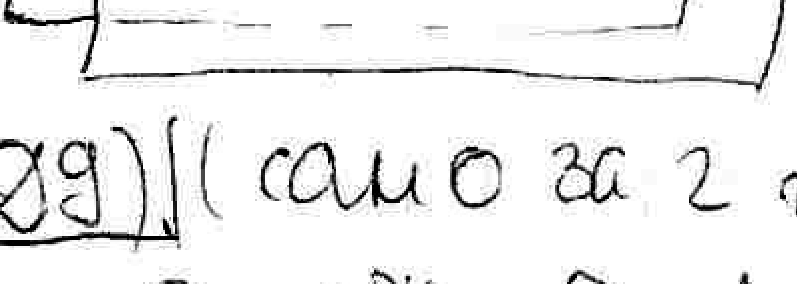
Нека $\varphi(x) = f(x,y_0)$ в/у $(x_0-\delta, x_0+\delta) \Rightarrow$



$\forall x \in (x_0-\delta, x_0+\delta) \Rightarrow \varphi(x) = f(x,y_0) \leq f(x_0,y_0) = \varphi(x_0)$
$\Rightarrow$ т. $x_0$ – лок. max за $\varphi(x)$
$\Rightarrow \varphi'(x_0) = \dfrac{\partial f(x_0,y_0)}{\partial x} = 0$ ($\exists$, з-то $\exists$ частна пр. отн. $x$)

Аналог. за $y$.

**Dy|** т. $(x_0,y_0)$: $\dfrac{\partial f(x_0,y_0)}{\partial x} = \dfrac{\partial f(x_0,y_0)}{\partial y} = 0 \Rightarrow$ т. $(x_0,y_0)$ – <u>стационарна точка</u>



**ДУ (ДУ)|** (само за 2 пром.!) Нека $f(x,y)$ е непр. заедно със своите ч. пр. до 2$^{ри}$ ред в ок. $B_\Delta(x_0,y_0)$ и $f'_x(x_0,y_0) = 0$, $f'_y(x_0,y_0) = 0$

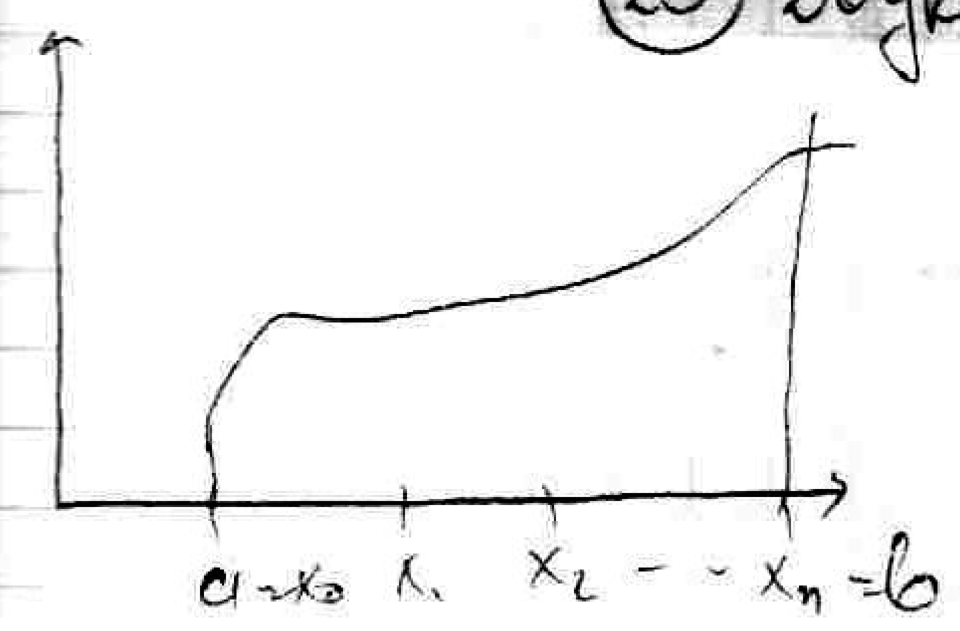
Нека $D(x_0,y_0) = f''_{xx}(x_0,y_0) \cdot f''_{yy}(x_0,y_0) - [f''_{xy}(x_0,y_0)]^2$. Тогава, ако:

1) $D(x_0,y_0) > 0 \Rightarrow$ т. $(x_0,y_0)$ – т. на лок. екстр. при това, ако:
1.1) $f''_{xx}(x_0,y_0) > 0$ – min    1.2) $f''_{xx}(x_0,y_0) < 0$ – max

2) $D(x_0,y_0) < 0 \Rightarrow (x_0,y_0)$ не е т. на лок. екстр.

3) $D=0$ – неопределеност

## (23) Двукратен интеграл – определение, свойства.




$f(x)$ – деф $[a,b]$

$\tau = \{x_i\}_{i=0}^{n}$, $a = x_0 < x_1 < \dots < x_n = b$

$\xi_i \in [x_{i-1}, x_i]$, $\xi = \{\xi_i\}_{i=1}^{n}$,

$\sigma_\tau(f,\xi) = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i$

$\Delta x_i = x_i - x_{i-1}$

$\delta_\tau = \max \Delta x_i$

$\int_a^b f(x) = \lim_{\delta_\tau \to 0} \sigma_\tau(f,\xi)$

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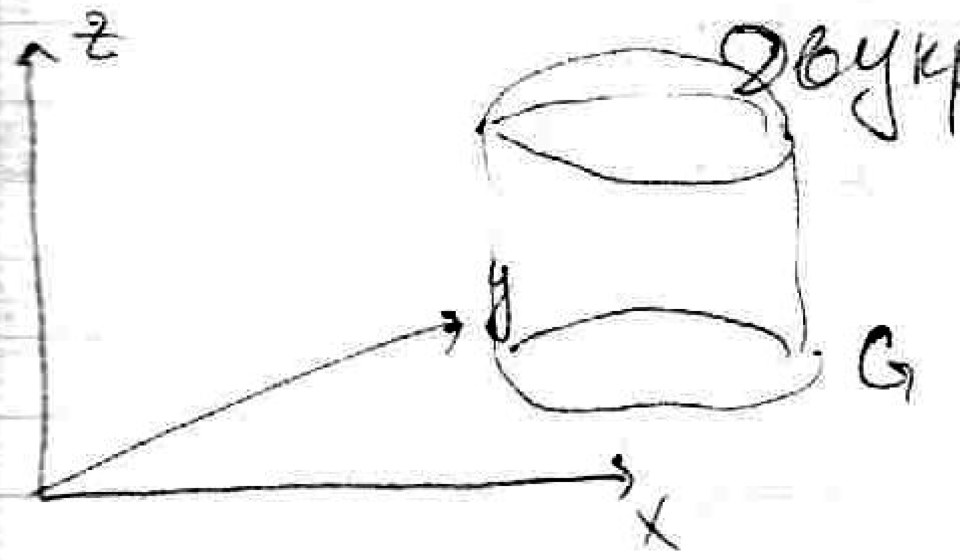
### Двукратно определени интеграли




Нека $\tau = \{G_i\}_{i=1}^{n}$:

1) $G_i \subset G$, $G_i$ – измеримо

2) $\bigcup_{i=1}^{n} G_i = G$

3) $G_i^\circ \cap G_j^\circ = \emptyset$, $\forall\, i,j = 1 \div n$, $i \neq j$

$\tau$ – разб. на $G$

$\forall\, i = 1 \div n$: $\xi_i(x_i,y_i) \in G(x_i,y_i) \in G_i$, $\xi = \{\xi_i\}_1^n$, $\sigma_\tau(f,\xi) = \sum_{i=1}^{n} f(x_i,y_i)\, m(G_i)$ ← сума на Риман

Деф $T \subset \mathbb{R}^2$: $d(T) = \max_{M,N \in T} d(M,N) = \max_{(x,y),(x',y') \in T} d((x,y),(x',y'))$ ← диаметър на $T$

$\delta_\tau = \max_{1 \le i \le n} d(G_i)$ – големина на разбиването

$\iint_G f(x,y)\,dx\,dy = \lim_{\delta_\tau \to 0} \sigma_\tau(f,\xi)$

Деф Казваме, че $f(x,y)$ е инт. в/у $G$, ако $\exists\, I \in \mathbb{R}$: $\forall\, \varepsilon > 0$,

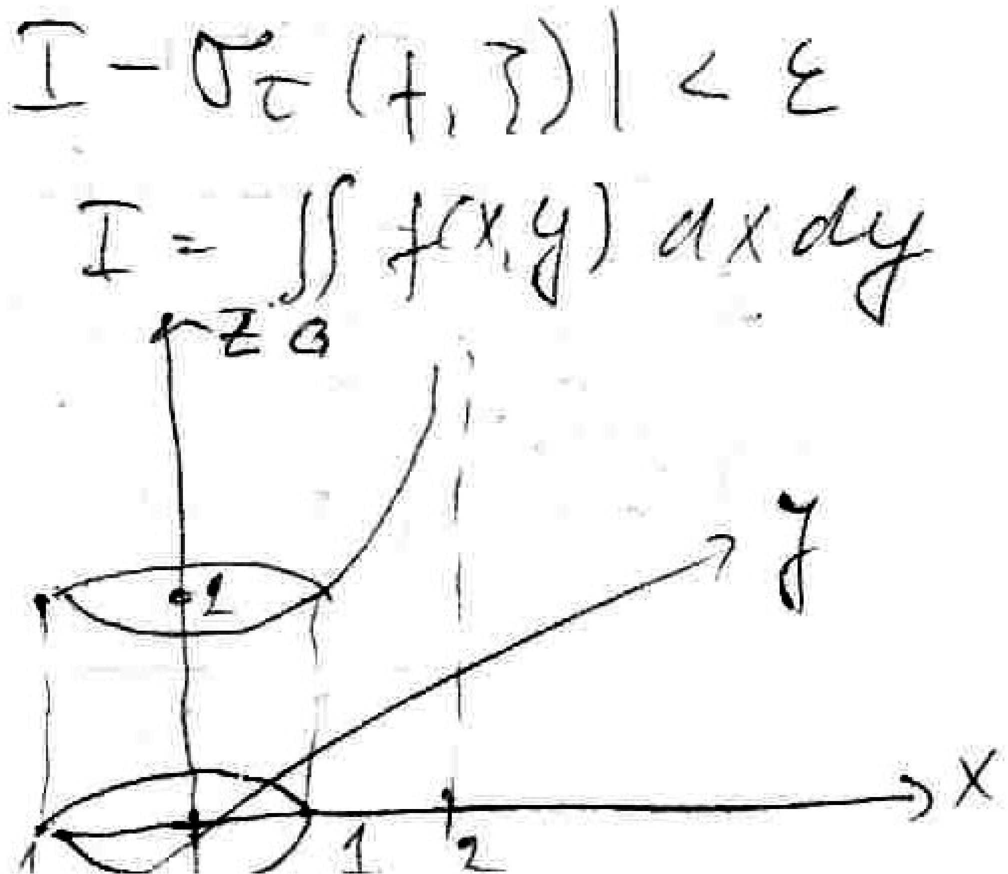
$\exists\, I \in \mathbb{R}$: $\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$: $\forall\, \tau = \{G_i\}_1^n$, $\delta_\tau < \delta$,

$\forall\, \xi = \{\xi_i\}_1^n$, $\xi_i \in G_i$, $(i = 1 \div n) \Rightarrow |I - \sigma_\tau(f,\xi)| < \varepsilon$

$I$ – двукр. инт. от $f(x,y)$ в/у $G$: $I = \iint f(x,y)\,dx\,dy$

Пример: $f(x,y) = \begin{cases} 1, & x^2+y^2 \le 1 \\ \frac{1}{2-x}, & x \in [1,2) \end{cases}$

$f(x,y)$ неогр $\Rightarrow$ не е инт.



**Деф** Нека $f(x,y)$ е огр. в/у $G$ – изм. в м-во. Казваме, че $f(x,y)$ е съществено огр. в/у $G$, ако $\exists\ E \subset G$, $E$ – Жорданова мярка нула $f(x,y)$ е огр. в/у $G \setminus E$.

**Деф** $f(x,y)$ – огр. в/у изм. м-во $G$. $\tau = \{G_i\}_{i}^{n}$ – разб. на $G$

$m_i = \inf\limits_{(x,y)\in G_i} f(x,y)$ , $M_i = \sup\limits_{(x,y)\in G_i} f(x,y)$

$s_\tau = \sum\limits_{i=1}^{n} m_i . m(G_i)$ – малка

$S_\tau = \sum\limits_{i=1}^{n} M_i . m(G_i)$ – голяма

сума на Дарбу

**Т (Кр. за интегр.)** $f(x,y)$ е инт. в/у изм. м-во $G \Leftrightarrow$ $\forall\ \varepsilon > 0, \exists\ \delta = \delta(\varepsilon) : \forall\ \tau = \{G_i\}_{i=1}^{n}, \delta_\tau < \delta \Rightarrow S_\tau < \varepsilon$

Д-во: (както при 1 пром.)

**Т** Ако $f(x,y)$ е непр. в/у комп. изм. м-во $G \Rightarrow f(x,y)$ е инт. в/у

Свойства: $G$ – измеримо:

1) $\iint\limits_G f(x,y)\,dx\,dy = S(G)$

2) $f$ и $g$ – инт. в/у $G \Rightarrow f+g$, $\lambda f$ – инт. в/у $G$, при това

$\iint\limits_G (\lambda f(x,y) + \mu g(x,y))\,dx\,dy = \lambda \iint\limits_G f(x,y)\,dx\,dy + \mu \iint\limits_G g(x,y)\,dx\,dy$

3) Ако $f(x,y) \geq 0$ в/у $G \Rightarrow \iint\limits_G f(x,y)\,dx\,dy \geq 0$
и интегр.

4) $G = G_1 \cup G_2$, $\{G_1, G_2\}$ – разб. на $G$, измерими ...

$\Rightarrow \iint\limits_G f(x,y)\,dx\,dy = \iint\limits_{G_1} f(x,y)\,dx\,dy + \iint\limits_{G_2} f(x,y)\,dx\,dy$

5) Ако $f$ е инт. в/у $G \Rightarrow |f|$ е инт. в/у $G$ и

$\left| \iint\limits_G f(x,y)\,dx\,dy \right| \leq \iint\limits_G |f(x,y)|\,dx\,dy$

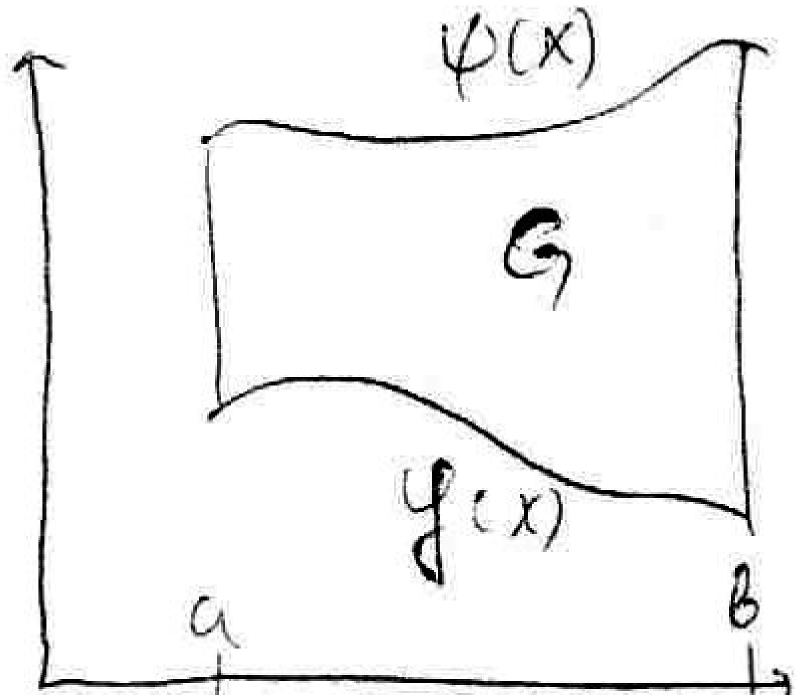
б) Ако $f$ е непр. в/у свърз. комп. изм. к-во $G \Rightarrow$

$\exists$ т. $(x_0, y_0)$: $\iint\limits_G f(x,y)\,dx\,dy = f(x_0, y_0) \cdot m(G)$

$f(x_0, y_0) = \dfrac{1}{m(G)} \iint\limits_G f(x,y)\,dx\,dy$ = ср. ст. на $f(x,y)$ в/у $G$

(24) Свеждане на двукр. инт. към повторен.

Смяна на променливите в двукратен интеграл.




Нека $G = \{(x,y): a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}$ – (криволинеен трапец)

и $\varphi, \psi$ – непр. в/у $[a,b]$ – ф-ции на $x$

Тh $\iint\limits_G f(x,y)\,dx\,dy = \underbrace{\int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right] dx}$

$F(x)$ – повторен интеграл

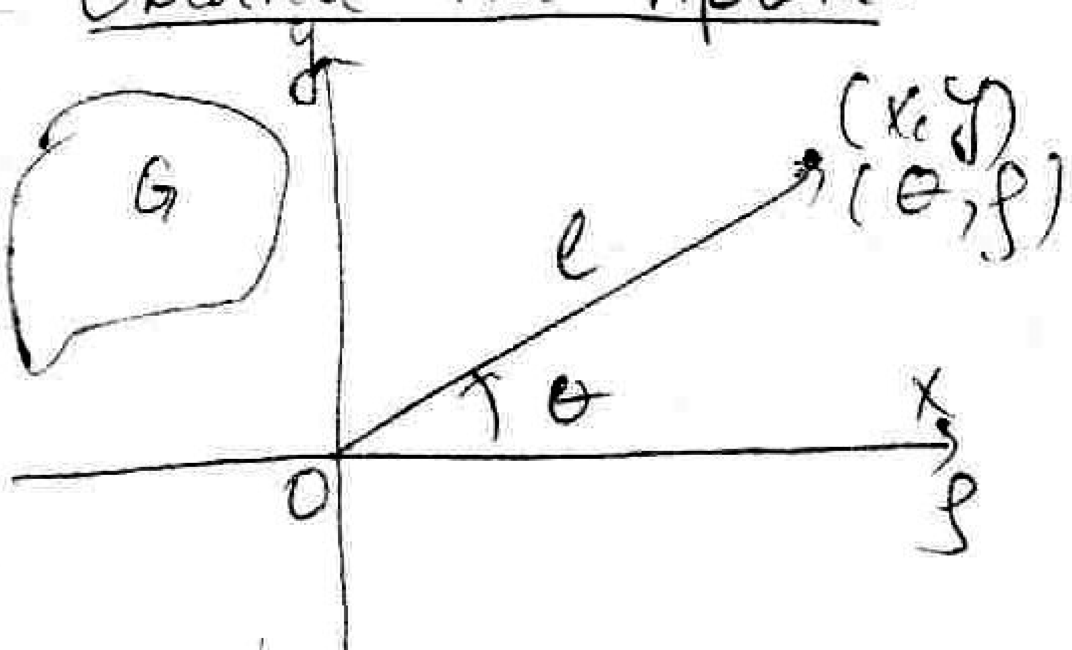
фигурата е измерима

$x$ – фикс. парам.

ф-та е отн. $y$, в инт. в границите

последично интегр. ф-ции на 1 пром

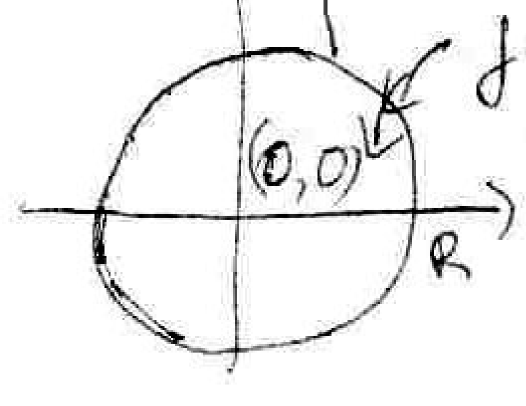
• Смяна на пром.




$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$, $\rho \ge 0$, $0 \le \theta \le 2\pi$

$$\boxed{\iint\limits_{G\,д.к.} f(x,y)\,dx\,dy = \iint\limits_{G\,п.с.} f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\theta\,d\rho}$$

декартова к.с. полярна с-ма




$f(x,y)$ в кръга деф.

$K: x^2 + y^2 \le R^2$

$K: 0 \le \theta \le 2\pi$
$0 \le \rho \le R$

$$\iint\limits_{K\,д.с.} f(x,y)\,dx\,dy = \iint\limits_{K\,п.с.} f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\theta\,d\rho =$$

$$= \int_0^{2\pi}\left[\int_0^R f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\theta\right] d\rho$$