15 | Производни от по-висок ред. Формула на Лайбниц

Д Нека $f(x)$ е дефинирана върху $U(x_0)$ и $\exists f'(x)$ за $\forall x \in U(x_0)$.

$(f'(x_0))' = f''(x_0)$ – втора производна на f в x_0

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

Ако съществува $f^{(n)}(x)$ в околност на $x_0 \Rightarrow \exists (f^{(n)}(x))' = f^{(n+1)}(x)$.

1) $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$

2) $f(x) = a^x \Rightarrow f^{(n)}(x) = a^x \ln^n x$

3) $f(x) = \sin x \Rightarrow f^{(n)}(x) = \sin\left(x + \frac{\pi}{2} n\right)$

$\sin' x = \cos x$

$\sin'' x = -\sin x$

$\sin''' x = -\cos x$

$\sin^{IV} x = \sin x$

4) $(\cos)^{(n)}(x) = \cos\left(x + \frac{\pi}{2} n\right)$

5) $f(x) = x^\alpha$

$f'(x) = \alpha x^{\alpha - 1}$

$f''(x) = \alpha . (\alpha - 1) . x^{\alpha - 2}$

. . . .

$f^{(n)}(x) = \alpha (\alpha - 1) \ldots (\alpha - n + 1) . x^{\alpha - n}$

6) $f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

. . .

$f^{(n)}(x) = \frac{(-1)^{n-1}}{x^n}$

Т Нека $f(x)$ и $g(x)$ имат $f^{(n)}(x)$ и $g^{(n)}(x)$ върху $(a, b]$.

$\Rightarrow$ 1) $(f \pm g)^{(n)}(x) = f^{(n)}(x) \pm g^{(n)}(x)$

2) $(\lambda f)^{(n)}(x) = \lambda f^{(n)}(x)$, $\lambda \in \mathbb{R}$

3) Формула на Лайбниц.

$$(f.g)^n = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) \cdot g^{(k)}(x)$$, където $f^0(x) = f(x)$, $g^0(x) = g(x)$

Доказателство:

1) индукция

$n=1$

$(f\pm g)' = f'(x) \pm g'(x)$ → вярно

Допускаме, че е вярно за някое k.

$(f\pm g)^{(k)}(x) = f^{(k)}(x) \pm g^{(k)}(x)$.

$(f\pm g)^{(k+1)} = ((f\pm g)^{(k)}(x))' = (f^{(k)}(x) \pm g^{(k)}(x))' = (f^{(k)}(x))' \pm (g^{(k)}(x))' =$

$= f^{(k+1)}(x) \pm g^{(k+1)}(x)$.

2) $(\lambda f)'(x) = \lambda f'(x) + f(x).\underset{0}{\lambda'} = \lambda f'(x)$

$(\lambda f)^{(k)}(x) = (\lambda) f^{(k)}(x)$ → допускаме

$(\lambda f)^{(k+1)}(x) = (\lambda f^{(k)})' = \lambda f^{(k+1)}(x)$.

3) $n=1$

$(f.g)'(x) = f'(x)\, g(x) + g'(x).f(x) = \sum_{k=0}^{1}\binom{1}{k} f^{(1-k)}(x).g^{(k)}(x)$ → вярно

Допускаме, че $(f.g)^{(m)}(x) = \sum_{k=0}^{m}\binom{m}{k} f^{(m-k)}(x).g^{(k)}(x)$

$(f.g)^{(m+1)}(x) = ((f.g)^{(m)}(x))' = \left(\sum_{k=0}^{m} f^{(m-k)}(x)\, g^{(k)}(x)\right)' =$

$= \sum_{k=0}^{m}\binom{m}{k}\left(f^{(m-k)}(x).g^{(k)}(x)\right)' = \sum_{k=0}^{m}\binom{m}{k} f^{(m+1-k)}(x)\, g^{(k)}(x) + \sum_{k=0}^{m}\binom{m}{k} f^{(m-k)}(x).g^{(k+1)}(x) =$

$= \binom{m}{0} f^{(m+1)}(x)\, g(x) + \sum_{k=1}^{m} f^{(m+1-k)}(x)\, g^{(k)}(x) + \sum_{k=0}^{m-1}\binom{m}{k} f^{(m-k)}(x)\, g^{(k+1)}(x) + \binom{m}{m} f(x) + g^{(m+1)}(x) =$

$= \binom{m+1}{0} f^{(m+1)}(x)\, g(x) + \sum_{p=1}^{m} f^{(m+1-p)}(x)\, g^{(p)}(x)\binom{m}{p} + \sum_{p=1}^{m}\binom{m}{p-1} f^{(m+1-p)}(x)\, g^{(p)}(x) + \binom{m+1}{m+1} f(x)\, g^{(m+1)}(x)$

$= \binom{m+1}{0} f^{(m+1)}(x)\, g(x) + \sum_{p=1}^{m}\left(\binom{m}{p} + \binom{m}{p-1}\right) f^{(m-p+1)}(x)\, g^{(p)}(x) + \binom{m+1}{m+1} f(x)\, g^{(m+1)}(x) =$

$= \sum_{p=0}^{m+1}\binom{m+1}{p} f^{(m+1-p)}(x)\, g^{(p)}(x)$