30 Суми на Дарбу. Критерий за интегруемост

$\boxed{D}$ Нека $f(x)$ е интегруема върху $[a,b]$, $\tau=\{x_i\}_{i=1}^{n}$ - разбиване на $[a,b]$.

$m_i=\inf\limits_{x\in[x_{i-1},x_i]} f(x)$; $M_i=\sup\limits_{x\in[x_{i-1},x_i]} f(x)$

$s_\tau=\sum\limits_{i=1}^{n} m_i\Delta x_i$ — малка } сума на Дарбу

$S_\tau=\sum\limits_{i=1}^{n} M_i\Delta x_i$ — голяма

$\Pi=\bigcup\limits_{i=1}^{n}[x_{i-1},x_i]\times[0,m_i]$

$\text{лице}(\Pi)=\sum\limits_{i=1}^{n}\text{лице}([x_{i-1},x_i]\times[0,m_i])=\sum\limits_{i=1}^{n} m_i\Delta x_i=s_\tau$

$S_\tau=\sum\limits_{i=1}^{n} M_i\Delta x_i$

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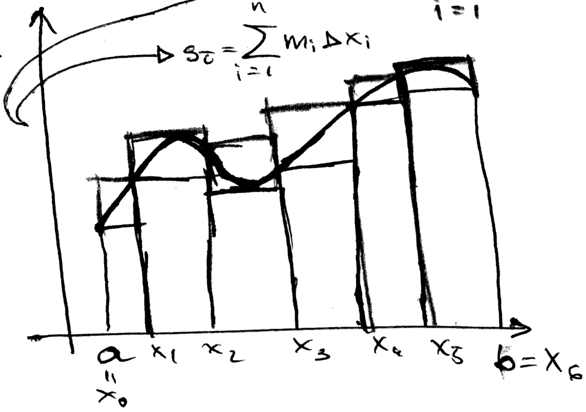


Функцията $f(x)$ е интегруема върху $[a,b]$

$\Downarrow$

Свойства:

1) $\forall\tau=\{x_i\}_{i=0}^{n}$

$\forall\xi=\{\xi_i\}_{i=1}^{n},\ \xi_i\in[x_{i-1},x_i],\ i=\overline{1,n}\Rightarrow$

$\Rightarrow s_\tau\le\sigma_\tau(f,\xi)\le S_\tau$

Доказателство:

$\forall\tau=\{x_i\}_{i=1}^{n},\ \xi=\{\xi_i\}_{i=1}^{n},\ \xi_i\in[x_{i-1},x_i],\ i=\overline{1,n}$

$M_i\ge f(\xi_i)\ge m_i,\ i=\overline{1,n}\quad |\cdot\Delta x_i$

$\Delta x_i m_i\le\Delta x_i f(\xi_i)\le M_i\Delta x_i\quad |\sum$

$\sum\limits_{i=1}^{n}\Delta x_i m_i\le\sigma_\tau(f,\xi)\le\sum\limits_{i=1}^{n} M_i\Delta x_i$

$\Downarrow$

$s_\tau\le\sigma_\tau(f,\xi)\le S_\tau$

2) $\forall\tau=\{x_i\}_{i=0}^{n}$, $s_\tau=\inf\limits_{\xi}\sigma_\tau(f,\xi)$; $S_\tau=\sup\limits_{\xi}\sigma(f,\xi)$

От свойство 1 при фиксирано $\tau=\{x_i\}_{i=1}^{n}$ имаме, че [redacted]

$^*\{\sigma_\tau(f,\xi);\ \xi=\{\xi_i\}_{i=1}^{n},\ \xi_i\in[x_{i-1},x_i]\ i=\overline{1,n}\}$ е ограничена отдолу $\Rightarrow$

$\Rightarrow\exists\inf\sigma_\tau(f,\xi)$ и s_τ е долна граница на $*$. Остава да докажем, че s_τ е $\inf$, т.е. $\forall\varepsilon>0\ \exists\ \xi_\varepsilon=\{\xi_{i\varepsilon}\}_{i=1}^{n}:\ \sigma_\tau(f,\xi_\varepsilon)<s_\tau+\varepsilon$

$\forall\varepsilon>0\Rightarrow\varepsilon_0=\dfrac{\varepsilon}{b-a}>0$ и разглеждаме $f(x)$ върху $[x_{i-1},x_i]$ $i=\overline{1,n}$

$m_i = \inf\limits_{[x_{i-1},x_i]} f(x) \Rightarrow$ За $\mathcal{E}_0 > 0 \quad \exists\, \xi_i^{\varepsilon} \in [x_{i-1}, x_i] : f(\xi_i^{\varepsilon}) < m_i + \mathcal{E}_0 \mid \Delta x_i$

$\Rightarrow f(\xi_i^{\varepsilon})\Delta x_i < m_i \Delta x_i + \mathcal{E}_0 \Delta x_i \quad \Big| \sum$

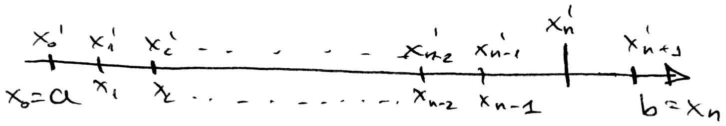
$$\sum_{i=1}^{n} f(\xi_i^{\varepsilon}) . \Delta x \leq \sum_{i=1}^{n} m_i \Delta x_i + \sum_{i=1}^{n} \mathcal{E}_0 \Delta x_i$$

$$\sigma_\tau(f, \xi^{\varepsilon}) \leq s_\tau + \mathcal{E}_0 \sum_{i=1}^{n} \Delta x_i = s_\tau + \mathcal{E} \Rightarrow s_\tau = \inf \sigma(f, \xi)$$

3) Нека $\tau < \tau' \Rightarrow s_\tau \leq s_{\tau'} \leq S_{\tau'} \leq S_\tau$

Доказателство:

$\tau = \{x_i\}_{i=0}^{n}$



$\tau' = \{x_i'\}_{i=0}^{n+1} = \tau \cup \{x_n'\} \Rightarrow \tau' > \tau$

$s_\tau \leq s_{\tau'}$ (аналогично за S_τ)

$m_i = \inf\limits_{[x_{i-1},x_i]} f(x)$; $m_i' = \inf\limits_{[x_{i-1},x_i]} f(x) \Rightarrow m_i = m_i'$, $i = \overline{1, n-1}$

$m'_{n+1} \geq m_n \geq m_n'$

$$m_n = \inf_{[x_{n-1},x_n]} f(x) \begin{cases} \leq m_n' = \inf\limits_{[x_{n-1},x_n]} f(x) \\ \leq m'_{n+1} = \inf\limits_{[x_n', x_{n+1}]} f(x) = \inf\limits_{[x_n', x_n]} f(x) \end{cases}$$

$$s_\tau = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n-1} m_i' \Delta x_i' + m_n \Delta x_n \leq \sum_{i=1}^{n} m_i' \Delta x_i' + m_n \Delta x_n' + m_n \Delta x_{n+1}' \leq$$

$$\leq \sum_{i=1}^{n} m_i' \Delta x_i' + m_n' \Delta x_n' + m_{n+1}' \Delta x_{n+1}' = \sum_{i=1}^{n+1} m_i' \Delta x_i' = s_\tau' \Rightarrow s_\tau \leq s_{\tau'}$$

4) $\forall \tau, \tau' \Rightarrow s_\tau \leq S_{\tau'}$

Доказателство:

$\tau'' = \tau \cup \tau' \Rightarrow \tau \subset \tau''$ и $\tau' \subset \tau''$

от 3) $\Rightarrow s_\tau \leq s_{\tau''} \leq S_{\tau''} \leq S_{\tau'} \Rightarrow s_\tau \leq S_{\tau'}$

5) $\exists \sup\limits_{\tau} s_\tau = \underline{I}$ и $\exists \inf\limits_{\tau} S_\tau = \overline{I} \Rightarrow s_\tau \leq \underline{I} \leq \overline{I} \leq S_\tau$

$\underbrace{\text{долен} \qquad\qquad\qquad \text{горен}}_{\text{интеграл на Дарбу}}$

Доказателство:

$\forall \tau, \tau' : s_{\tau'} \leq S_\tau$

Фиксираме $\tau \Rightarrow \forall \tau' \Rightarrow S_\tau \geq s_{\tau'} \Rightarrow \{S_\tau : \tau\}$ е ограничено отдолу $\Rightarrow$

$\Rightarrow \exists \inf\limits_{\tau} S_\tau = \overline{I} \Rightarrow s_{\tau'} \leq \overline{I} \;\; \forall \tau' \Rightarrow \{s_{\tau'} : \tau'\}$ е ограничено отгоре $\Rightarrow$

$\Rightarrow \exists \sup\limits_{\tau} s_\tau = \underline{I} \Rightarrow \underline{I} \leq \overline{I}$

$\boxed{\text{Т}}$ Критерий за интегруемост

Нека $f(x)$ е ограничена върху $[a,b]$. Функция $f(x)$ е интегруемо по Риман върху $[a,b] \Leftrightarrow \forall \varepsilon>0 \ \exists \delta_{(\varepsilon)}>0 : \forall \tau=\{x_i\}_{i=0}^{n}, \delta_\tau<\delta \Rightarrow S_\tau - s_\tau<\varepsilon$

Доказателство:

$\Rightarrow$) $f(x)$ е интегруема в $[a,b] \Rightarrow \exists I\in\mathbb{R} : \forall\varepsilon>0 \ \exists\delta_{(\varepsilon)}>0:$

$\forall \tau=\{x_i\}_{i=0}^{n}, \delta_\tau<\delta \Rightarrow \forall \xi=\{\xi_i\}_{i=1}^{n}, \xi_i\in[x_{i-1},x_i], i=\overline{1,n}$

$\Rightarrow |I-\sigma_\tau(f,\xi)|<\varepsilon \Rightarrow I-\varepsilon<\sigma_\tau(f,\xi)<I+\varepsilon \Rightarrow$

$\Rightarrow I-\varepsilon \le \inf_\tau \sigma_\tau(f,\xi) \le \sup_\tau \sigma_\tau(f,\xi) \le I+\varepsilon$

$I-\varepsilon \le \underset{s_\tau}{\shortparallel} \le S_\tau \le I+\varepsilon \Rightarrow S_\tau - s_\tau \le \varepsilon$

$\Leftarrow$) $\forall\varepsilon>0 \ \exists\delta_{(\varepsilon)}>0 : \forall\tau=\{x_i\}_{i=0}^{n} \ \delta_\tau<\delta \Rightarrow S_\tau - s_\tau<\varepsilon$

$\forall\tau: \ s_\tau \le \underline{I} \le \overline{I} \le S_\tau$

$\Rightarrow 0 \le \overline{I} - \underline{I} \le S_\tau - s_\tau < \varepsilon$

$\forall\varepsilon>0 \Rightarrow 0\le \overline{I}-\underline{I}<\varepsilon \Rightarrow \overline{I}-\underline{I}=0 \Rightarrow \overline{I}=\underline{I}=I$

$s_\tau \le I \le S_\tau \quad (1)$

$\tau \Rightarrow \sigma_\tau(f,\xi), \ \xi=\{\xi_i\}_{i=1}^{n}, \ \xi_i\in[x_{i-1},x_i], i=\overline{1,n}$

$$
\begin{aligned}
&s_\tau \le \sigma_\tau(f,\xi) \le S_\tau \\
+\ &-S_\tau \le -I \le -s_\tau
\end{aligned}
$$

$\shortparallel$

$-(S_\tau - s_\tau) \le \sigma_\tau(f,\xi) - I \le S_\tau - s_\tau$

$\Rightarrow |\sigma_\tau(f,\xi)-I| \le S_\tau - s_\tau < \varepsilon \Rightarrow \forall \xi=\{\xi_i\}_{i=1}^{n}, \xi_i\in[x_{i-1},x_i] \ i=\overline{1,n}$

$\Rightarrow \exists I\in\mathbb{R} : \forall\varepsilon>0 \ \exists\delta_{(\varepsilon)}>0 : \forall\tau=\{x_i\}_{i=0}^{n} \ \delta_\tau<\delta \ \forall\xi=\{\xi_i\}_{i=1}^{n}, \xi_i\in[x_{i-1},x_i]$

$\Rightarrow |I-\sigma_\tau(f,\xi)|<\varepsilon \Rightarrow f(x)$ е интегруема върху $[a,b]$

$\boxed{\text{Следствие 1}}$ Ако $f(x)$ е интегруема върху $[a,b]$, $\underline{I}=\overline{I}=I \Leftrightarrow$

$\Leftrightarrow \displaystyle\int_a^b f(x)dx = \sup_\tau s_\tau = \inf_\tau S_\tau$

$\boxed{D}$ Нека $f(x)$ е ограничена върху $X \subset \mathbb{R}$. Колебание на $f(x)$ върху $[a,b]$ се нарича числото

$$\omega(f) = \omega_X(f) = M - m, \text{ където } M = \sup_X f(X),\ m = \inf_X f(x)$$

$\boxed{Тв}$ $f(x)$ е ограничена върху $[a,b] \Rightarrow \omega(f) = \sup_{[a,b]} |f(x') - f(x'')|$.

Нека $f(x)$ е ограничена върху $[a,b]$ и $\tau = \{x_i\}_{i=0}^{n}$ е разбиване на $[a,b]$.

$$\omega_{[x_{i-1},x_i]}(f) = \omega_i(f) = M_i - m_i \Rightarrow M_i = \sup_{[x_{i-1},x_i]} f(x);\ m_i = \inf_{[x_{i-1},x_i]} f(x)$$

$$\Rightarrow S_\tau - s_\tau = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} \omega_i(f)\Delta x$$

$\boxed{T}$ Критерий за интегруемост

Нека $f(x)$ е ограничена върху $[a,b]$. $f(x)$ е интегруема върху $[a,b] \Leftrightarrow \forall \varepsilon > 0 : \forall \tau = \{x_i\}_{i=1}^{n},\ \delta_\tau < \delta \Rightarrow \sum_{i=1}^{n} \omega_i(f)\Delta x_i < \varepsilon$