13. Правила за намиране на производни, свързани с аритметичните действия с функции. Производна на сложна функция.

Т1 Нека $f(x)$ и $g(x)$ са дефинирани в $U(x_0)$ и $\exists f'(x_0)$ и $g'(x_0)$.

$\Rightarrow$ 1) $\exists (f+g)'(x_0) = f'(x_0) + g'(x_0)$;

2) $\exists (f-g)'(x_0) = f'(x_0) - g'(x_0)$;

3) $\exists (f.g)'(x_0) = f'(x_0)g(x_0) + f(x_0).g'(x_0)$

4) при $g(x_0) \neq 0$

$$\exists (f/g)'(x_0) = \frac{f'(x_0).g(x_0) - f(x_0).g'(x_0)}{g^2(x_0)}.$$

Доказателство:

1), 2) $$\frac{\Delta(f \pm g)}{\Delta x} = \frac{(f \pm g)(x_0+\Delta x) - (f \pm g)(x_0)}{\Delta x} = \frac{f(x_0+\Delta x) \pm g(x_0+\Delta x) - (f(x_0) \pm g(x_0))}{\Delta x} =$$

$$= \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \pm \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x} = \frac{\Delta f}{\Delta x} \pm \frac{\Delta g}{\Delta x}$$

$$\exists ? \lim_{\Delta x \to 0} \frac{\Delta(f \pm g)}{\Delta x} \longrightarrow \lim_{\Delta x \to 0}\left(\frac{\Delta f}{\Delta x} + \frac{\Delta g}{\Delta x}\right) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = f'(x_0) + g'(x_0).$$

3) $$\frac{\Delta(f.g)}{\Delta x} = \frac{(f.g)(x_0+\Delta x) - (f.g)(x_0)}{\Delta x} = \frac{f(x_0+\Delta x).g(x_0+\Delta x) - f(x_0).g(x_0)}{\Delta x} =$$

$$= \frac{[f(x_0+\Delta x).g(x_0+\Delta x) - f(x_0)g(x_0+\Delta x)] + [f(x_0).g(x_0+\Delta x) - f(x_0).g(x_0)]}{\Delta x} =$$

$$= \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} g(x_0+\Delta x) + f(x_0)\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x} = \frac{\Delta f}{\Delta x} g(x_0+\Delta x) + \frac{\Delta g}{\Delta x} f(x_0).$$

$$\lim_{\Delta x \to 0} \frac{\Delta(f.g)}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\Delta f}{\Delta x} g(x_0+\Delta x) + \frac{\Delta g}{\Delta x} f(x_0)\right) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \lim_{\Delta x \to 0} g(x_0+\Delta x) + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} \lim_{\Delta x \to 0} f(x_0) =$$

$$= f'(x_0).g(x_0) + g'(x_0).g(x_0).$$

4) $$\frac{\Delta(f/g)}{\Delta x} = \frac{(f/g)(x_0+\Delta x) - (f/g)(x_0)}{\Delta x} = \frac{\frac{f(x_0+\Delta x)}{g(x_0+\Delta x)} - \frac{f(x_0)}{g(x_0)}}{\Delta x} =$$

$$= \frac{1}{\Delta x} \cdot \frac{f(x_0+\Delta x).g(x_0) - f(x_0).g(x_0+\Delta x)}{g(x_0+\Delta x)g(x_0)} = \frac{1}{\Delta x} \cdot \frac{f(x_0+\Delta x).g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x_0+\Delta x)}{g(x_0+\Delta x)g(x_0)}.$$

$$= \frac{1}{g(x_0).g(x_0+\Delta x)}\left(\frac{g(x_0).(f(x_0+\Delta x) - f(x_0))}{\Delta x} - \frac{f(x_0)(g(x_0+\Delta x) - g(x_0))}{\Delta x}\right) =$$

$$= \frac{1}{g(x_0)g(x_0+\Delta x)}\left(\frac{\Delta f}{\Delta x}g(x_0) - \frac{\Delta g}{\Delta x}\cdot f(x_0)\right)$$

$$\lim_{\Delta x \to 0} \frac{\Delta(f/g)}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{1}{g(x_0)g(x_0+\Delta x)}\cdot\left(\frac{\Delta f}{\Delta x}g(x_0) - f(x_0)\frac{\Delta g}{\Delta x}\right)\right) =$$

$$= \frac{1}{g(x_0)\lim\limits_{\Delta x \to 0} g(x_0+\Delta x)}\left(\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}\cdot g(x_0) - f(x_0)\lim_{\Delta x \to 0}\frac{\Delta g}{\Delta x}\right) =$$

$$= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

$\boxed{T_2}$ Нека сложната функция $F = (f\circ g)(x)$ е дефинирана така:

$$U(x_0) \xrightarrow{g} U(u_0) \xrightarrow{f} \mathbb{R}$$

$$\underbrace{\phantom{U(x_0) \to U(u_0) \to \mathbb{R}}}_{F(x) = (f\circ g)(x) = f(g(x))},$$ където: $g(x_0) = u_0$, $\exists g'(x_0)$ и $f'(u_0)$

$\Rightarrow \exists F'(x_0) = f'(u_0)g'(x_0) = f'(g(x_0))\,g'(x_0)$

Доказателство:

дефиниране $\varphi(u) = \begin{cases} \dfrac{f(u)-f(u_0)}{u-u_0} - f'(u_0) & \to u \neq u_0 \\ 0 & \to u = u_0 \end{cases}$

$\varphi(u)$ - непрекъсната в u_0

$$\lim_{u\to u_0}\varphi(u) = \lim_{u\to u_0}\left(\frac{f(u)-f(u_0)}{u-u_0} - f'(u_0)\right) = f'(u_0) - f'(u_0) = 0 = \varphi(u_0)$$

$u \neq u_0 \Rightarrow f(u) - f(u_0) = (f'(u_0) + \varphi(u)) \cdot (u - u_0)$

$u = g(x)$; $u_0 = g(x_0)$: $\underbrace{f(g(x)) - f(g(x_0))}_{\Delta F = F(x) - F(x_0)} = (f'(u_0) + \varphi(g(x)))\cdot(g(x) - g(x_0))$

$$\frac{\Delta F}{\Delta x} = \frac{F(x)-F(x_0)}{x-x_0} = \frac{f(g(x)) - f(g(x_0))}{x-x_0} = \frac{(f'(u_0)+\varphi(g(x)))\cdot(g(x)-g(x_0))}{x-x_0} =$$

$$= (f'(u_0) + \varphi(g(x)))\frac{\Delta g}{\Delta x}$$

$$\lim_{x\to x_0}\frac{\Delta F}{\Delta x} = \lim_{x\to x_0}\left(f'(u_0) + \varphi(g(x))\right)\frac{\Delta g}{\Delta x} = \left(f'(u_0) + \lim_{x\to x_0}\varphi(g(x))\right)\cdot\lim_{x\to x_0}\frac{\Delta g}{\Delta x} =$$

$$= f'(u_0)\cdot g'(x_0) = f'(g(x_0))\,g'(x_0).$$