14 Производни на основните елементарни функции

1) $(c)' = 0$;

2) $(x)' = 1$;

3) $(x^2)' = 2x$;

4) $(x^n)' = nx^{n-1}, \quad n \in \mathbb{N}$;

$$\frac{\Delta x^n}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x} = \frac{\cancel{x^n} + \binom{n}{1} x^{n-1}\Delta x + \binom{n}{2} x^{n-2}\Delta x^2 + \dots + \Delta x^n - \cancel{x^n}}{\Delta x} = nx^{n-1} + \Delta x . P(\Delta x).$$

$$\lim_{\Delta x \to 0} \frac{\Delta x^n}{\Delta x} = \lim_{\Delta x \to 0}\left(nx^{n-1} + \Delta x P(\Delta x)\right) = n . x^{n-1}$$

5) $(\sin x)' = \cos x$

$$\frac{\Delta \sin x}{\Delta x} = \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{2\sin\frac{\Delta x}{2} \cdot \cos\frac{2x+\Delta x}{2}}{\Delta x} = \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\left(x + \frac{\Delta x}{2}\right)$$

$$\lim_{\Delta x \to 0} \frac{\Delta \sin x}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\left(x + \frac{\Delta x}{2}\right)\right) = \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\Delta x/2} \cdot \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) = \cos x$$

6) $(\cos x)' = -\sin x$

$$(\cos x)' = \left(\sin\left(\pi/2 - x\right)\right)' = \cos\left(\pi/2 - x\right) . (-1) = -\sin x$$

7) $(\operatorname{tg} x)' = 1/\cos^2 x$

8) $(\operatorname{cotg} x)' = -1/\sin^2 x$

} доказват се, като се представят като частни

9) $(\ln x)' = 1/x$

$$\frac{\Delta \ln x}{\Delta x} = \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \frac{1}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) = \frac{1}{x} \cdot \frac{x}{\Delta x} \cdot \ln\left(1 + \Delta x/x\right) =$$

$$= \frac{1}{x} . \ln\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$$\lim_{x \to 0} (1+x)^{1/x} = e \to \lim_{\Delta x \to 0} \frac{\Delta \ln x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{x} . \ln\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} = \frac{1}{x} \ln e = \frac{1}{x}$$

10) $(\log_a x)' = \dfrac{1}{x \ln a} \to (\log_a x)' = \left(\dfrac{\ln x}{\ln a}\right)' = \dfrac{1}{\ln a} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln a}$

11) $(a^x)' = a^x \ln a$

$$f(x) = a^x \to \ln a^x = \ln(f(x)) = x \ln a$$

$$\left(\ln(f(x))\right)' = (x \ln a)' \to \frac{1}{f(x)} . f'(x) = \ln a \to f'(x) = f(x) . \ln a$$

12) $(e^x)' = e^x$

13) $(x^{\alpha})' = \alpha . x^{\alpha-1}$, $\alpha \in \mathbb{R}$ при $x > 0$

$f(x) = x^{\alpha} \Rightarrow \ln f(x) = \ln x^{\alpha} \rightarrow \ln f(x) = \alpha \ln x$

$(\ln f(x))' = (\alpha \ln x)'$

$\frac{1}{f(x)} \cdot f'(x) = \frac{\alpha}{x} \quad f'(x) = \frac{\alpha}{x} \cdot x^{\alpha} = \alpha . x^{\alpha-1}$

$f^{-1}(f(x)) \equiv x \quad f(\underbrace{f^{-1}(x)}_{y}) \equiv x \Rightarrow f'_y (f^{-1})'(x) = (x)'_x$

$f(f^{-1}(x)) \equiv x$

$f'_y (f^{-1})'_x = 1 \Rightarrow (f^{-1})'_x = \frac{1}{f'_y} = \frac{1}{f'(f(x))} \Rightarrow (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$

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14) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$f'(x) = (\arcsin x)' = \frac{1}{(\sin f(x))'_{f(x)}} = \frac{1}{\cos(f(x))} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}$

15) $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

16) $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

17) $(\operatorname{arccotg} x)' = \frac{-1}{1+x^2}$

$f(x) = \operatorname{arctg} x \Rightarrow f'(x) = \frac{1}{(\operatorname{tg} f(x))'_{f(x)}} = \cos^2 f(x) = \frac{1}{1+\operatorname{tg}^2 f(x)} = \frac{1}{\cancel{1+\operatorname{arctg}} \operatorname{tg}^2(\operatorname{arctg} x)} =$

$= \frac{1}{x^2+1}$