5 | Свойства на граница, свързани с аритметичните действия

T Нека функциите $f(x)$ и $g(x)$ са дефинирани поне върху $\mathring{U}(x_0)$ и $\exists \lim_{x\to x_0} f(x) = A$ и $\lim_{x\to x_0} g(x) = B$. Тогава:

1) $\exists \lim_{x\to x_0} (f+g)(x) = A+B$;

2) $\exists \lim_{x\to x_0} (f-g)(x) = A-B$;

3) $\exists \lim_{x\to x_0} (f.g)(x) = A.B$;

4) $\exists \lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = A/B$, ако $B \neq 0$.

1 Нека функцията $f(x)$ е дефинирана върху $\mathring{U}(x_0)$. Ако $\exists c>0$: $\forall \varepsilon>0, \exists \delta=\delta(\varepsilon)>0 : \forall x\in \mathring{U}_\delta(x_0) \rightarrow |f(x)-A| < c.\varepsilon \Rightarrow \exists \lim_{x\to x_0} f(x) = A$.

(1) Доказателство: Фиксираме $\varepsilon>0$.

$\varepsilon' = \frac{\varepsilon}{c} > 0$ $\exists \delta = \delta(\varepsilon') = \delta(\varepsilon) > 0$:

$\forall x \in \mathring{U}_\delta(x_0) \rightarrow |f(x) - A| < c.\varepsilon' = c.\frac{\varepsilon}{c} = \varepsilon \Rightarrow A = \lim_{x\to x_0} f(x)$

(T) Доказателство:

1) $\lim_{x\to x_0}(f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$

Нека фиксираме $\varepsilon>0$:

$A = \lim_{x\to x_0} f(x) \Rightarrow \exists \delta_1 = \delta_1(\varepsilon)>0 : \forall x\in \mathring{U}_{\delta_1}(x_0) \rightarrow |f(x)-A|<\varepsilon$

$B = \lim_{x\to x_0} g(x) \Rightarrow \exists \delta_2 = \delta_2(\varepsilon)>0 : \forall x\in \mathring{U}_{\delta_2}(x_0) \rightarrow |g(x)-B|<\varepsilon$

$\delta = \min\{\delta_1, \delta_2\} \rightarrow \forall x \in \mathring{U}_\delta(x) \rightarrow \left| \begin{array}{l} |f(x)-A|<\varepsilon \\ |g(x)-B|<\varepsilon \end{array} \right. \Rightarrow$

$\Rightarrow |(f+g)(x) - (A+B)| = |f(x)+g(x) - (A+B)| = |f(x) - A + g(x) - B| \leq$

$\leq |f(x)-A| + |g(x)-B| < 2.\varepsilon$; $\forall x\in \mathring{U}_\delta(x_0)$

Т.е. $\forall x \in \mathring{U}_\delta(x_0)$: $|(f+g)(x) - (A+B)| < 2.\varepsilon \overset{1}{\Rightarrow} \exists \lim_{x\to x_0}(f+g)(x) = A+B$

3) $\lim_{x\to x_0}(f.g)(x) = A.B$

$\lim_{x\to x_0} f(x) = A$ и $\lim_{x\to x_0} g(x) = B \Rightarrow$

$\exists \delta_1 > 0$: $g(x)$ е ограничена върху $\mathring{U}_{\delta_1}(x_0) \Rightarrow \exists M>0 : |g(x)| \leq M$ $\forall x\in \mathring{U}_{\delta_1}(x)$

Нека фиксираме $\varepsilon>0$: $\exists \delta_2 = \delta_2(\varepsilon)>0 : \forall x\in \mathring{U}_{\delta_2}(x_0) \rightarrow |f(x)-A|<\varepsilon$

$\exists \delta_3 = \delta_3(\varepsilon)>0 : \forall x\in \mathring{U}_{\delta_3}(x_0) \rightarrow |g(x)-B|<\varepsilon$

Нека $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0 \Rightarrow \begin{cases} |g(x)| \le M \\ |f(x) - A| < \varepsilon \\ |g(x) - B| < \varepsilon \end{cases}$ за $\forall x \in \mathring{U}_\delta(x_0)$.

Разглеждаме $(f.g)(x)$ върху $\mathring{U}_\delta(x_0)$.

$$|(f.g) - A.B| = |f(x).g(x) - A.B| = |f(x)\,g(x) - A.g(x) + A.g(x) - A.B| =$$

$$= |g(x)(f(x) - A) + A(g(x) - B)| \le |g(x)|\,|f(x) - A| + |g(x) - B|\,|A| <$$

$$< M\varepsilon + |A|.\varepsilon = \varepsilon\underbrace{(M + |A|)}_{const} \xRightarrow{1} \exists \lim_{x \to x_0} (f.g)(x) = AB$$

4) $\lim\limits_{x \to x_0} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{A}{B}$ при $B \neq 0$

$\boxed{Л_2}$ Ако $\lim\limits_{x \to x_0} g(x) = B \neq 0$, то $\exists \mathring{U}_\delta(x_0)$: $\dfrac{1}{g(x)}$ е ограничена върху $\mathring{U}_\delta(x_0)$

Л Доказателство: $B = \lim\limits_{x \to x_0} g(x)$. Нека $\varepsilon = |B|/2 > 0 \Rightarrow$

$\Rightarrow \exists \delta = \delta(\varepsilon) > 0 : \forall x \in \mathring{U}_\delta(x_0) \Rightarrow$

$$\Rightarrow \frac{|B|}{2} < |g(x)| \Rightarrow \frac{1}{|g(x)|} < \frac{2}{|B|} \Rightarrow \left|\frac{1}{g(x)}\right| < \frac{2}{|B|} \Rightarrow \frac{1}{g(x)} \text{ е ограничена}$$

Т Доказателство:

4) т.к. $\lim\limits_{x \to x_0} g(x) = B \neq 0 \xRightarrow{Л_2} \exists \delta > 0 \ \exists M > 0 : \forall x \in \mathring{U}_\delta(x_0) \rightarrow \left|\dfrac{1}{g(x)}\right| < M$ (1).

От това, че $\lim\limits_{x \to x_0} f(x) = A$ и $\lim\limits_{x \to x_0} g(x) = B$, то $\forall \varepsilon > 0 \ \exists \delta_1 = \delta_1(\varepsilon) > 0$:

$\forall x \in \mathring{U}_{\delta_1}(x_0) \rightarrow |f(x) - A| < \varepsilon$ (2) и $\exists \delta_2 = \delta_2(\varepsilon) > 0$:

$\forall x \in \mathring{U}_{\delta_2}(x_0) \rightarrow |g(x) - B| < \varepsilon$ (3)

Нека $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0 \ \forall x \in \mathring{U}_\delta(x_0) \Rightarrow \begin{cases} \left|\frac{1}{g(x)}\right| < M \\ |f(x) - A| < \varepsilon \\ |g(x) - B| < \varepsilon \end{cases}$

Разглеждаме: $\left|\left(\dfrac{f}{g}\right)(x) - \dfrac{A}{B}\right| = \left|\dfrac{f(x)}{g(x)} - \dfrac{A}{B}\right| = \left|\dfrac{B.f(x) - A.g(x)}{Bg(x)}\right| =$

$$= \left|\frac{Bf(x) - AB + BA - A.g(x)}{B.g(x)}\right| = \left|\frac{B(f(x) - A) - A(g(x) - B)}{Bg(x)}\right| \le$$

$$\le \frac{|B|\,|f(x) - A| + |A|\,|g(x) - B|}{|B|\,|g(x)|} \le \frac{M}{|B|}\left(|B|\,|f(x) - A| + |A|\,|g(x) - B|\right) <$$

$$< \frac{M}{|B|}(|B|\varepsilon + |A|\varepsilon) = \frac{M(|B| + |A|)}{|B|}.\varepsilon = c.\varepsilon \Rightarrow \exists \lim_{x \to x_0}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$$

$\boxed{T}$ Нека сложната функция $(F \circ f)(x)$ е зададена по следния начин:

$$\begin{tikzcd} \mathring{U}(x_0) \arrow[r, "f"] \arrow[rr, bend right, "(F\circ f)(x) = F(f(x))"'] & \mathring{U}(y_0) \arrow[r, "F"] & \mathbb{R} \end{tikzcd}$$

Ако $\exists \lim\limits_{x \to x_0} f(x) = y_0$ и $\exists \lim\limits_{y \to y_0} F(y) = L \Rightarrow \lim\limits_{x \to x_0} (F \circ f)(x) = L$

Доказателство: Нека $\varepsilon > 0$ и т.к. $\exists \lim\limits_{y \to y_0} F(y) = L$, то за $\varepsilon > 0$ $\exists \delta_1 = \delta_1(\varepsilon) > 0$: $\forall y \in \mathring{U}_{\delta_1}(y_0) \Rightarrow |F(y) - L| < \varepsilon$ (1)

Т.к. $\exists \lim\limits_{x \to x_0} f(x) = y_0$, то за $\delta_1 > 0$ $\exists \delta = \delta(\delta_1) = \delta(\varepsilon) > 0$: $\forall x \in \mathring{U}_{\delta}(x_0) \Rightarrow |f(x) - y_0| < \delta_1$ (2)

От (1) и (2) $\Rightarrow |(F \circ f)(x) - L| = |F(f(x)) - L| < \varepsilon \Rightarrow$

$\Rightarrow \lim\limits_{x \to x_0} (F \circ f)(x) = L$