35 | Лице на равнинна фигура. Обем на ротационно тяло

D Множеството $X \subset \mathbb{R}^2$ се нарича ограничено, ако може да се включи в кръг с даден радиус.

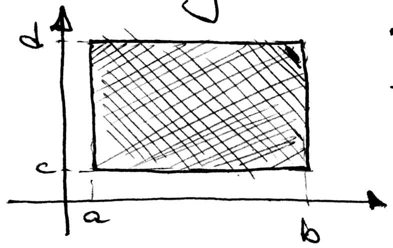
D Всяко ограничено множество $\overline{X} \subset \mathbb{R}$ се нарича фигура.

D Окръжност – $\{M \in \mathbb{R}^2 : |MO| = r\}$.

D Кръг – $\{M \in \mathbb{R}^2 : |MO| \leq r\}$.

D Клетка в $\mathbb{R}$ се нарича множество от вида

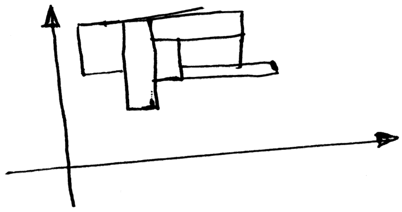
$$\Pi = \{(x,y) : a \prec x \prec b,\ c \prec y \prec d;\ \prec \in \{<, \leq\}\} = \langle a, b \rangle \times \langle c, d \rangle.$$



D Лице на клетка Π: $S(\Pi) = (b-a)(d-c)$.

D Клетъчно множество в $\mathbb{R}$ – всяко множество $K = \bigcup_{i=1}^{n} \Pi_i$, така че $\Pi_i^\circ \cap \Pi_j^\circ = \varnothing$. Π° – вътрешност на клетка

$$\Pi^\circ = \{(x,y) : a < x < b,\ c < y < d\}.$$



D Лице на клетъчно множество $K = \bigcup_{i=1}^{n} \Pi_i$.

$$S(K) = \sum_{i=1}^{n} S(\Pi_i)$$

1) $K_1 \cap K_2 = \varnothing \Rightarrow S(K_1 \cup K_2) = S(K_1) + S(K_2)$

2) $K_1 \subseteq K_2 \Rightarrow S(K_1) \leq S(K_2)$

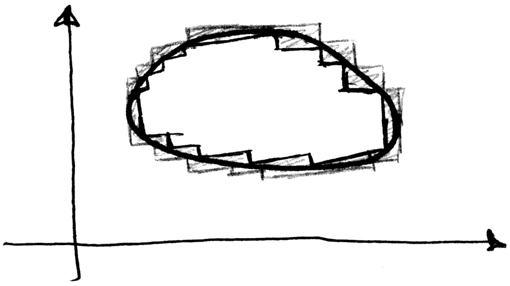
3) K – клетъчно множество; $x \in \mathbb{R}^2$

$$\left.\begin{array}{l} x + K = \{x + y : y \in K\} \\ A = (x_1, y_1) \quad \lambda A(\lambda x_1, \lambda y_1) \\ + \ B = (x_2, y_2) \\ \hline (x_1 + x_2, y_1 + y_2) \end{array}\right\} \Rightarrow S(x+K) = S(K)$$

D Равнинната фигура $G \subset \mathbb{R}^2$ се нарича измерима, ако $\forall \varepsilon > 0\ \exists$ клетъчни множества K и k:

1) $k \subset G \subset K$

2) $S(K) - S(k) < \varepsilon$



D Нека G е измеримо множество в $\mathbb{R}^2$. Лице на G наричаме такова число $S(G)$, че $\forall K, k$ - клетъчни множества: $k \subset G \subset K \Rightarrow$
$\Rightarrow S(k) \leq S(G) \leq S(K)$.

T Всяко измеримо множество G има единствено лице. При това
$S(G) = \sup\limits_{k \subset G} S(k) = \inf\limits_{K \supset G} S(K)$.

Доказателство:

$\forall k, K : k \subset G \subset K \Rightarrow S(k) \leq S(K)$

Фиксираме K:

$$\left.\begin{array}{l} \{S(k) : k \subset G\} \\ \{S(K) : K \subset G\} \end{array}\right\} \Rightarrow \exists \sup\limits_{k \subset G} S(k) \leq S(K) \Rightarrow \exists \inf\limits_{K \supset G} S(K) \geq \sup\limits_{k \subset G} S(k) \Rightarrow$$

$$\Rightarrow \inf\limits_{K \supset G} (S(K)) - \sup\limits_{k \subset G} S(k) \geq 0$$

$$S(K) - S(k) \geq \inf\limits_{K \supset G} S(K) - \sup\limits_{k \subset G} S(k)$$

G - измеримо $\Rightarrow \forall \varepsilon > 0 : \exists k, K :$ 1) $k \subset G \subset K$
2) $S(K) - S(k) < \varepsilon$

$$\Rightarrow 0 \leq \inf\limits_{K \supset G} S(K) - \sup\limits_{k \subset G} S(k) \leq S(K) - S(k) < \varepsilon \quad \forall \varepsilon > 0 \Rightarrow$$

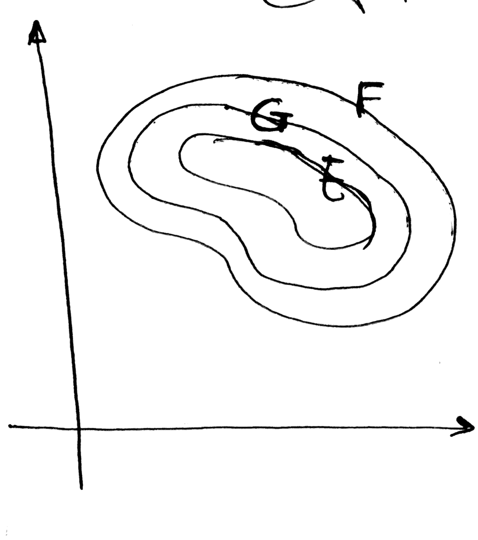
$$\Rightarrow \inf S(K) = \sup S(k)$$

$$S(k) \leq S(G) \leq S(K) \quad \forall k, K : k \subset G \subset K \Rightarrow$$

$$\Rightarrow \inf\limits_{K \supset G} S(K) \geq S(G) \geq \sup\limits_{k \subset G} S(k) \Rightarrow \exists\, S(G) = \sup\limits_{k \subset G} S(k) = \inf\limits_{K \supset G} S(K)$$

T Критерий за измеримост

Равнинна фигура G е измерима $\Leftrightarrow$
$\Leftrightarrow \forall \varepsilon > 0 \; \exists$ измерими множества E и F: 1) $E \subset G \subset F$
2) $S(F) - S(E) < \varepsilon$



Доказателство:

$\Rightarrow$) G - измеримо $\Rightarrow \forall \varepsilon > 0$
$\exists k, K$ клетъчни множества:
1) $k \subset G \subset K$
2) $S(K) - S(k) < \varepsilon$

Всяко клетъчно множество е измеримо.

$\Leftarrow$) $\forall \varepsilon > 0 \to \varepsilon/3$

$\exists E$ и F – измерими: 1) $E \subset G \subset F$

2) $S(F) - S(E) < \varepsilon/3$

E – измеримо $\Rightarrow S(E) = \sup\limits_{k \subset E} S(k) \Rightarrow \varepsilon/3 > 0 \ \exists k_\varepsilon \subset E: \ S(E) - \frac{\varepsilon}{3} < S(k_\varepsilon)$

$\Rightarrow S(E) - S(k_\varepsilon) < \varepsilon/3$

F – измеримо $\Rightarrow S(F) = \inf\limits_{K \supset F} S(K) \Rightarrow \frac{\varepsilon}{3} > 0 \ \exists K_\varepsilon \subset E: \ S(K_\varepsilon) < S(F) + \frac{\varepsilon}{3}$

$\Rightarrow S(K_\varepsilon) - S(F) < \varepsilon/3$

$S(K_\varepsilon) - S(k_\varepsilon) = (S(K_\varepsilon) - S(F)) + (S(F) - S(E)) + (S(E) - S(k_\varepsilon)) < \varepsilon$

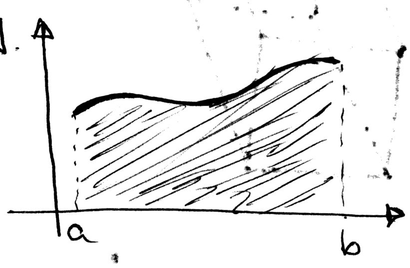
$\Rightarrow \forall \varepsilon > 0: \exists K, k$ – клетъчни: 1) $k \subset G \subset K$

2) $S(K) - S(k) < \varepsilon$

$\Rightarrow S(G) = \sup\limits_{k \subset G} S(k) = \inf\limits_{K \supset G} S(K)$

$\boxed{D}$ Нека $f(x) \geq 0$ е непрекъсната върху $[a,b]$.

$D = \{(x,y) : x \in [a,b], y \in [0, f(x)]\}$ се нарича криволинеен трапец.

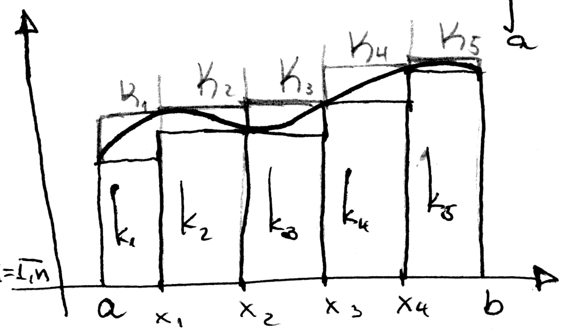


$\boxed{T}$ D е криволинеен трапец. D е измеримо множество и $S(D) = \int_a^b f(x)dx$.

Доказателство:

$f(x)$ е непрекъсната $\Rightarrow f(x)$ е интегруема върху $[a,b] \Rightarrow \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon)$:

$\forall \tau = \{x_i\}_{i=0}^{n} \ \delta_\tau < \delta \Rightarrow S_\tau - s_\tau < \varepsilon$



$k_i = [x_{i-1}, x_i] \times [0, m_i] = \{(x,y) : x \in [x_{i-1}, x_i], y \in [0, m_i]\} \ i = \overline{1,n}$

$S_{k_i} = \Delta x_i . m_i$

$s_\tau = \sum\limits_{i=1}^{n} m_i \Delta x_i = \sum\limits_{i=1}^{n} S(k_i) = S(k) : k = \bigcup\limits_{i=1}^{n} k_i$

$K_i = [x_{i-1}, x_i] \times [0, M_i] = \{(x,y) : x \in [x_{i-1}, x_i], y \in [0, M_i]\} \ i = \overline{1,n}$

$S(K_i) = \Delta x_i . M_i$

$S_\tau = \sum\limits_{i=1}^{n} \Delta x_i M_i = \sum\limits_{i=1}^{n} S(K_i) = S(K) : K = \bigcup\limits_{i=1}^{n} K_i$

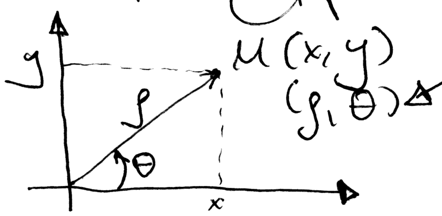
$\downarrow$

1) $k \subset D \subset K$

2) $S(K) - S(k) = S_\tau - s_\tau < \varepsilon \Rightarrow D$ е измеримо $\Rightarrow$

$\Rightarrow S(D) = \sup S(k) = \sup s_\tau = \int_a^b f(x)dx$

$\boxed{D}$ Полярни координати



$\rightarrow x = \rho\cos\theta;\ y = \rho\sin\theta$

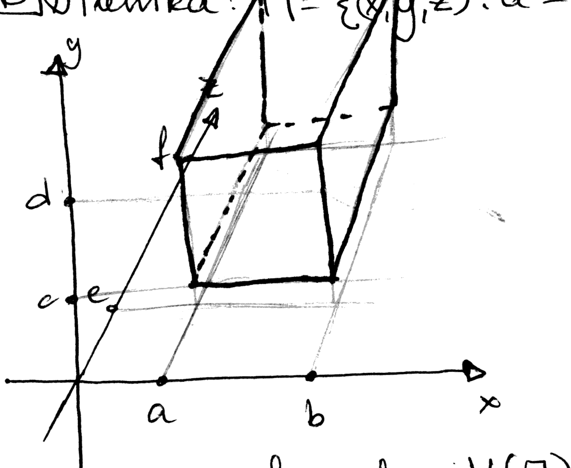
$\boxed{T}$ Нека $\rho = \rho(\theta)$, $\theta \in [\alpha, \beta]$, ρ е непрекъсната.

$D = \{(\rho, \theta) : \alpha \le \theta \le \beta;\ 0 \le \rho \le \rho(\theta)\} \Rightarrow S(D) = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\theta)\,d\theta$

$\boxed{D}$ Всяко множество $X \subset \mathbb{R}^3$, което може да се вмести в кълбо, се нарича тяло.

$\boxed{D}$ Кълбо: $\{M \subset \mathbb{R}^3 : |MO| = r\}$, $O \in \mathbb{R}^3$

$\boxed{D}$ Клетка: $\Pi = \{(x, y, z) : a \le x \le b, c \le y \le d, e \le z \le f\} = \langle a, b\rangle \times \langle c, d\rangle \times \langle e, f\rangle$.



$\boxed{D}$ Обем на клетка $V(\Pi) = (b-a)(d-c)(f-e)$

$\boxed{D}$ Клетъчно тяло $T = \bigcup_{i=1}^{n} \Pi_i : \Pi_i^\circ \cap \Pi_j^\circ = \emptyset$

$V(T) = \sum_{i=1}^{n} V(\Pi_i)$

$\boxed{D}$ Нека $\Omega \subset \mathbb{R}^3$. Казваме, че Ω е измеримо, ако

$\forall \varepsilon > 0\ \exists\, t, T$-клетъчни множества: 1) $t \subset \Omega \subset T$

2) $V(T) - V(t) < \varepsilon$

~~D Нека Ω ⊂ R³. Казваме, че~~

$\boxed{D}$ Ако Ω е измеримо тяло, то обем на Ω наричаме

$V(\Omega$: $\forall t, T$-клетъчни множества: $t \subset \Omega \subset T \Rightarrow V(t) \le V(\Omega) \le V(T)$

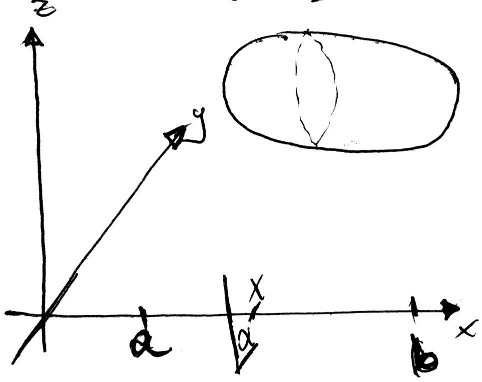
$\boxed{T}$ Ако Ω е измеримо тяло, $\exists V(\Omega) = \sup V(t) = \inf V(T)$.

$\boxed{T}$ Тялото Ω е измеримо $\Leftrightarrow$

$\forall \varepsilon > 0$ $\exists$ измерими тела Ω_1 или Ω_2:

1) $\Omega_1 \subset \Omega \subset \Omega_2$

2) $V(\Omega_2) - V(\Omega_1) < \varepsilon$



$\alpha \parallel yOz$

$\alpha \cap Ox = x$

$\forall x \in [a,b] \Rightarrow S(x) = S(T(x))$

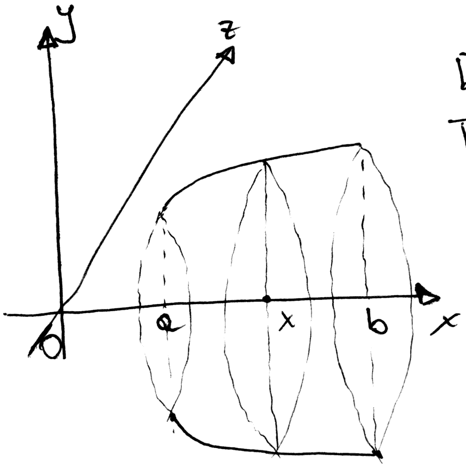
$S(x)$ — лице на сечението, $T(x) = \alpha_x \cap \Omega$

$S: [a,b] \to \mathbb{R}$

$S = S(x) =$ лицето $(T(x))$

$\boxed{T}$ Ако $S(x)$ е непрекъсната, $V(T) = \int_a^b S(x)\,dx$.

$\boxed{T}$ Нека $f(x)$ е неотрицателна, непрекъсната функция върху $[a,b]$.



$D = \{(x,y) : x \in [a,b],\ y \in [0, f(x)]\}$

$T_x = \alpha_x \cap T$

$S(x) = S(T_x) = \pi f^2(x)$

$V(T) = \int_a^b \pi f^2(x)\,dx = \pi \int_a^b f^2(x)\,dx$