

$$e_k \in a_k + \lambda_{k1} e_1 + \dots + \lambda_{k, k-1} e_{k-1}$$

$$\left. \begin{array}{l} 1) e_k \in \ell(a_1, \dots, a_k) \\ a_k \in \ell(e_1, \dots, e_k) \end{array} \right\} \text{следствие} \\ \dim(\ell(e_1, \dots, e_k)) = k$$

$$2) e_1, \dots, e_k - \text{базис на } \ell(e_1, \dots, e_k)$$

$$\Rightarrow e_1, \dots, e_k - \text{ЛНЗ}$$

$$e_k \neq 0$$

$$3) \cancel{e_i \in \ell(e_1, \dots, e_k)} e_i \perp e_k \quad i \in [1, k-1]$$

Следствие: \forall крайномерно ЕП има ортонормиран базис

Детерминанта на Грэм

$$E - \text{ЕП}, a_1, \dots, a_k \in E$$

$$\Gamma(a_1, \dots, a_k) = \begin{vmatrix} (a_1, a_1) & (a_1, a_2) & \dots & (a_1, a_k) \\ (a_2, a_1) & (a_2, a_2) & \dots & (a_2, a_k) \\ \vdots & \vdots & \ddots & \vdots \\ (a_k, a_1) & (a_k, a_2) & \dots & (a_k, a_k) \end{vmatrix}$$

$$\boxed{T} \quad \Gamma(a_1, \dots, a_k) \geq 0 \quad \wedge \quad \Gamma(a_1, \dots, a_k) = 0 \Leftrightarrow a_1, \dots, a_k \text{ са}$$

ЛЗ.

Доказ-во:

$$\textcircled{1} a_1, \dots, a_k - \text{ЛЗ}$$

$$\Rightarrow \exists \lambda_1, \dots, \lambda_k \neq 0, \dots, 0 : \lambda_1 a_1 + \dots + \lambda_k a_k = 0$$

$$\left\{ \begin{array}{l} \lambda_1 (a_1, a_1) + \dots + \lambda_k (a_1, a_k) = 0 \\ \lambda_1 (a_2, a_1) + \dots + \lambda_k (a_2, a_k) = 0 \\ \vdots \\ \lambda_1 (a_k, a_1) + \dots + \lambda_k (a_k, a_k) = 0 \end{array} \right.$$

после a_1
...
 a_k

$$\left(\begin{array}{l} \lambda_1 (a_k, a_1) + \dots + \lambda_k (a_k, a_k) = 0 \end{array} \right.$$

\Rightarrow ЛЗ с неизвестни λ_i , когато има ненулево решение $\Rightarrow \text{rk}(A) < k \Rightarrow \det(A) = 0$
и $\det(A) = \Gamma = 0$

② a_1, \dots, a_k - ЛНЗ $\Rightarrow \dim(\ell(a_1, \dots, a_k)) = k \Rightarrow k$ -мерно подпространство на n -мерное $\mathbb{R}^n \Rightarrow$
 $\Rightarrow \exists e_1, \dots, e_k$ - ~~базис на~~ ортонормированный базис на

$$\ell(a_1, \dots, a_k) \Rightarrow a_s = \alpha_{s1}e_1 + \dots + \alpha_{sk}e_k \quad s=1, \dots, k$$

$$(a_s, a_t) = \alpha_{s1}\alpha_{t1} + \alpha_{s2}\alpha_{t2} + \dots + \alpha_{sk}\alpha_{tk} = \sum_{i=1}^k \alpha_{si}\alpha_{ti}$$

$$\Gamma(a_1, \dots, a_k) = \begin{pmatrix} \sum_{i=1}^k \alpha_{1i}\alpha_{1i} & \sum_{i=1}^k \alpha_{1i}\alpha_{2i} & \dots & \sum_{i=1}^k \alpha_{1i}\alpha_{ki} \\ \sum_{i=1}^k \alpha_{2i}\alpha_{1i} & \sum_{i=1}^k \alpha_{2i}\alpha_{2i} & \dots & \sum_{i=1}^k \alpha_{2i}\alpha_{ki} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^k \alpha_{ki}\alpha_{1i} & \dots & \dots & \sum_{i=1}^k \alpha_{ki}\alpha_{ki} \end{pmatrix} =$$

умножение на матрицы

$$= \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k1} & \alpha_{k2} & \dots & \alpha_{kk} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{k1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1k} & \alpha_{2k} & \dots & \alpha_{kk} \end{pmatrix}$$

Δ

$$= \Delta \cdot \Delta^t = \Delta^2 \geq 0$$

Предположим на Δ совпадают координатные на a_1, \dots, a_k , которые с ЛНЗ $\Rightarrow \Delta \neq 0 \Rightarrow \Delta^2 > 0 = \Gamma(a_1, \dots, a_k)$

Следствие:

$$\Gamma(a_1, a_2) \geq 0$$

$$\Gamma(a_1, a_2) = \frac{(a_1, a_1)(a_1, a_2)}{(a_2, a_1)(a_2, a_2)} \geq 0$$

$$(a_1, a_1)(a_2, a_2) - (a_1, a_2)^2 \geq 0$$

$$(a_1, a_2)^2 \leq |a_1|^2 |a_2|^2$$

$$\boxed{|(a_1, a_2)| \leq |a_1| |a_2|} \rightarrow \text{Неравенство на Коши-Буняковского / НКБ!}$$

НКБ: a_1, \dots, a_n - ортонормирован базис

$$a_1 = \alpha_1 e_1 + \dots + \alpha_n e_n$$

$$b = \beta_1 e_1 + \dots + \beta_n e_n$$

$$|\alpha_1 \beta_1 + \alpha_2 \beta_2 + \dots + \alpha_n \beta_n| \leq \sqrt{\alpha_1^2 + \dots + \alpha_n^2} \sqrt{\beta_1^2 + \dots + \beta_n^2}$$

~~(a_1, a_2)~~

Дефиниране на косинус

$$a_1, a_2 \neq 0$$

$$\frac{(a_1, a_2)}{|a_1| |a_2|} \in [-1, 1]$$

$$\text{Опр. } \cos(a_1, a_2) = \frac{(a_1, a_2)}{|a_1| |a_2|}$$

Алгебрина Дефиниция

Следствие: Неравенство на триъгълника

$$\begin{aligned} |a+b|^2 &= (a+b, a+b) = (a, a) + 2(a, b) + (b, b) \leq |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

$$|a+b|^2 \leq (|a| + |b|)^2 \Rightarrow |a+b| \leq |a| + |b|$$

D Нека U е подпространство на E -ЕП.

Дуално / ортогонално н.во

$$U^\perp = \{x \in E \mid x \perp a, \forall a \in U\}$$

1) U^\perp също е подпространство на E

$$x, y \in U^\perp \quad a \in U \quad x \perp a, y \perp a$$

$$(\lambda x + \mu y, a) = \lambda(x, a) + \mu(y, a) = 0$$

$$\Rightarrow \lambda x + \mu y \perp a \Rightarrow \lambda x + \mu y \in U^\perp$$

T Ако E е крайномерно ЕП и U е подпространство,

тогава $E = U \oplus U^\perp$

↓
директна сума-когато $U \cap U^\perp = \{0\}$ е нулевото подпространство

Доказателство:

① U -ната базис, т.е. $U = \{0\} \Rightarrow U^\perp = E \Rightarrow U = \{0\}$

$$\Rightarrow E = U \oplus \{0\}$$

② $U \neq \{0\} \Rightarrow e_1, \dots, e_k$ -ордонормиран базис

Допълваме с e_{k+1}, \dots, e_n и получаваме e_1, \dots, e_n -ордонормиран базис на E .

$$W = \ell(e_{k+1}, \dots, e_n)$$

$$\forall x \in W, a \in U$$

$$x = \mu_{k+1}e_{k+1} + \dots + \mu_n e_n \rightarrow (0, 0, \dots, 0, \mu_{k+1}, \dots, \mu_n)$$

$$a = \alpha_1 e_1 + \dots + \alpha_k e_k \rightarrow (\alpha_1, \dots, \alpha_k, 0, \dots, 0)$$

$$(x, a) = 0 = 0\alpha_1 + \dots + 0\alpha_k + 0\mu_{k+1} + \dots + 0\mu_n \Rightarrow$$

$$\Rightarrow x \in U^\perp \Rightarrow W \subset U^\perp$$

2) $y \in U^\perp \Rightarrow y = \beta_1 e_1 + \dots + \beta_k e_k + \beta_{k+1} e_{k+1} + \dots + \beta_n e_n$

$$y \perp e_1 \Rightarrow (e_1, y) = \beta_1 = 0 \text{ аналог. } \beta_k = 0$$

$$\Rightarrow y = \beta_{k+1} e_{k+1} + \dots + \beta_n e_n \Rightarrow$$

$$\Rightarrow \forall y \in W \Rightarrow W \subset U^\perp \Rightarrow W = U^\perp$$

$$\underbrace{e_1 \dots e_k}_{\text{Basis of } U}, \underbrace{e_{k+1} \dots e_n}_{\text{Basis of } U^\perp}$$

Basis of U

Basis of U^\perp

$$\Rightarrow \ell(e_1 \dots e_k) \cap \ell(e_{k+1} \dots e_n) = U \cap U^\perp = \{0\}$$

$$\Rightarrow U + U^\perp = E$$

$$x \in E \Rightarrow x = \underbrace{j_1 e_1 + \dots + j_k e_k}_{u \in U} + \underbrace{j_{k+1} e_{k+1} + \dots + j_n e_n}_{u^\perp \in U^\perp}$$

Corollary

$$\dim U^\perp = \dim E - \dim U$$

Corollary

$$(U^\perp)^\perp = U$$