

16.12.2014

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Алгебра - I - лекция

[...] Умножение на детерминанти. [...]

Лема: Флека $\varphi: F^n \rightarrow F^n$

e_1, \dots, e_n - стандартен базис

$a_1, \dots, a_n \in F^n$, тогава $\det(\varphi(a_1), \dots, \varphi(a_n)) =$
 $= \det A \cdot \det(a_1, \dots, a_n)$, където A - матрица на φ
 в e_1, \dots, e_n

Док - до:

I cr: $\det A = 0 \Rightarrow \varphi(e_1) \dots \varphi(e_n)$ са лз
 $\Rightarrow r(\varphi) = \dim \operatorname{Im} \varphi < n$

$\varphi(a_1), \dots, \varphi(a_n) \in \operatorname{Im} \varphi$ са лз

$\Rightarrow \det(\varphi(a_1), \dots, \varphi(a_n)) = 0$

II cr: $\det A \neq 0$

$$\frac{\det(\varphi(a_1), \dots, \varphi(a_n))}{\det(A)} = f(a_1, \dots, a_n)$$

$$\begin{aligned} f(a_1, \dots, \lambda a_i' + \mu a_i'', \dots, a_n) &= \frac{1}{\det A} \det(\varphi(a_1), \dots, \varphi(\lambda a_i' + \mu a_i''), \dots, \varphi(a_n)) \\ &= \frac{1}{\det A} (\lambda \det(\varphi(a_1), \dots, \varphi(a_i'), \dots, \varphi(a_n)) + \\ &\quad \mu \det(\varphi(a_1), \dots, \varphi(a_i''), \dots, \varphi(a_n))) = \lambda f(a_1, \dots, a_i', \dots, a_n) + \\ &\quad + \mu f(a_1, \dots, a_i'', \dots, a_n) \end{aligned}$$

f -многолинейна

$i \neq j$

$$a_i = a_j = b$$

$$f(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = \frac{\det(\varphi(a_1), \dots, \varphi(b), \dots, \varphi(b), \dots, \varphi(a_n))}{\det A}$$

$= 0 \Rightarrow f$ е антисимметрична

$$f(e_1, \dots, e_n) = \frac{\det(\varphi(e_1), \dots, \varphi(e_n))}{\det A} = \frac{\det A}{\det A} = 1$$

$$\Rightarrow f(a_1, \dots, a_n) = \det(a_1, \dots, a_n) = \frac{\det(\varphi(a_1), \dots, \varphi(a_n))}{\det A}$$

Th. 11 Ако A, B са $n \times n$ матрици

$$\det(AB) = \det A \cdot \det B$$

Док - во: _____

F^n, A, B - матрици на $\varphi, \psi: F^n \rightarrow F^n$.

$C = AB$ матрица на $\varphi \circ \psi$

$$\det C = \det(\varphi \circ \psi(e_1), \dots, \varphi \circ \psi(e_n)) = \det A \cdot \det(\varphi(e_1), \dots,$$

$$\varphi(e_n)) = \det A \cdot \det B \cdot \underbrace{\det(e_1, \dots, e_n)}_1 = \underline{\underline{\det A \cdot \det B}}$$

$\text{Hom}(V, V)$

$\varphi \circ \psi$ - няма комутативност, но има асоциативност

$$(\varphi \circ \psi) \circ \tau = \varphi \circ (\psi \circ \tau)$$

id - идентитет, то е действителен ^{линеен} оператор,

$$\text{id}: V \rightarrow V: \text{id}(x) = x$$

$$\varphi \circ \text{id} = \varphi = \text{id} \circ \varphi$$

Опр.: Оператор $\varphi: V \rightarrow V$ е обратим, ако $\exists \psi: V \rightarrow V$:

$$\varphi \circ \psi = \psi \circ \varphi = \text{id}$$

Свойства на обратими оператори:

① Ако φ е обратим, то $\exists ! \psi: \varphi \circ \psi = \psi \circ \varphi = \text{id}$

Доказ-во: _____

Допускаме, че ψ_1 и ψ_2 използват условията

$$\psi_1 = \psi_1 \circ \text{id} = \psi_1 \circ (\varphi \circ \psi_2) = (\psi_1 \circ \varphi) \circ \psi_2 = \text{id} \circ \psi_2 = \psi_2$$

$$\Rightarrow \psi_1 = \psi_2 = \varphi^{-1}$$

② Ако φ е обратим, φ^{-1} също е обратим и $(\varphi^{-1})^{-1} = \varphi$

$$\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = \text{id}$$

③ Ако φ е обратим и α е const., $\alpha \neq 0$, $\alpha \in F \Rightarrow$
 $\Rightarrow (\alpha \varphi)^{-1} = \alpha^{-1} \varphi^{-1}$

$$(\alpha \varphi) \circ (\alpha^{-1} \varphi^{-1}) = ~~\alpha \alpha^{-1}~~ (\alpha \cdot \alpha^{-1}) \varphi \circ \varphi^{-1} = 1 \cdot \text{id} = \text{id}$$

④ Ако φ и ψ са обратими, то $\varphi \circ \psi$ е обратима

$$\underline{(\varphi \circ \psi)^{-1} = \psi^{-1} \circ \varphi^{-1}}$$

$$\begin{aligned}
 (\varphi \circ \varphi) \circ (\varphi^{-1} \circ \varphi^{-1}) &\stackrel{\text{ассоц.}}{=} \varphi \circ (\varphi \circ \varphi^{-1}) \circ \varphi^{-1} = \\
 &= \varphi \circ \text{id} \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = \text{id}
 \end{aligned}$$

$$\underline{(\varphi^{-1} \circ \varphi^{-1}) \circ (\varphi \circ \varphi) = \text{id}}$$

Th. // Если V конечномерно пр-во и $\varphi: V \rightarrow V$ ^{линейный оператор}.

Следующие условия эквивалентны:

1) φ - обратим оператор

2) $\text{Im } \varphi = V \Leftrightarrow (\text{r}(\varphi) = \dim V)$

3) $\text{Ker } \varphi = \{0\} \Leftrightarrow (\text{d}(\varphi) = 0)$

4) Если e_1, \dots, e_n - базис на V , то $\varphi(e_1), \dots, \varphi(e_n)$ също - базис на V

Док-во:

1) \Rightarrow 2)

φ - обратим оператор

$$\Rightarrow \exists \varphi^{-1}: V \rightarrow V$$

$$x \in V \quad \varphi(\varphi^{-1}(x)) = x$$

$$\Rightarrow x \in \text{Im } \varphi \Rightarrow \text{Im } \varphi = V$$

$$\Rightarrow \text{r}(\varphi) = \dim V$$

2) \Rightarrow 3) $\text{Im} = V, \text{r}(\varphi) = \dim V$

$$\Rightarrow \text{d}(\varphi) = \dim \text{Ker}(\varphi) = \dim V - \text{r}(\varphi) = 0 \Rightarrow \text{Ker } \varphi = \{0\}$$

$$3) \Rightarrow 1) \ker \varphi = \{0\}$$

$$\Rightarrow x \neq y \Rightarrow \varphi(x) \neq \varphi(y)$$

$$\varphi(x-y) \neq 0$$

$$\Rightarrow \operatorname{Im} \varphi = V \Rightarrow \varphi \text{ е сюръективен} \Rightarrow \varphi \text{ е обратим}$$

$$\Rightarrow \exists \psi: V \rightarrow V$$

$$\varphi(a) = x \Rightarrow \psi(x) = a$$

$$\varphi(b) = y \Rightarrow \psi(y) = b$$

$$\varphi(\lambda a + \mu b) = \lambda \varphi(a) + \mu \varphi(b) = \lambda x + \mu y$$

$$\Rightarrow \varphi(\lambda x + \mu y) = \lambda a + \mu b = \lambda \psi(x) + \mu \psi(y)$$

$$\Rightarrow \varphi \text{ е линейен оператор}$$

$$\varphi \circ \psi = \psi \circ \varphi = \operatorname{id}$$

$$\Rightarrow \varphi \text{ е обратим}$$

$$2) \Leftrightarrow 4)$$

$$\varphi \text{-обратим} \Rightarrow \operatorname{Im} \varphi = V, \text{ ако } e_1, \dots, e_n \text{ базис на } V$$

$$\operatorname{Im} \varphi = \langle \varphi(e_1), \dots, \varphi(e_n) \rangle = V$$

$$\Rightarrow \varphi(e_1), \dots, \varphi(e_n) \text{ базис на } V$$

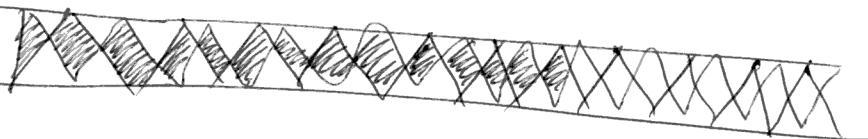
$$\text{Ако } \varphi \text{ изобразява базис в базис} \Rightarrow \varphi(e_1), \dots, \varphi(e_n) \text{ - базис}$$

$$\Rightarrow \operatorname{Im} \varphi = V$$

Контрпример: Когато пространството е безкрайномерно.

$$V = \mathbb{R}[x] \quad \partial: V \rightarrow V \quad \partial(f) = f'$$

Обратимы матрицы.



$$A \in M_{n \times n}(F)$$

$$E \cdot A = A \cdot E = A$$

$(AB)C = A(BC)$ - асоциативност, но няма комутативност

Опр. 1 Матрицата A е обратима, ако \exists матрица

$$B: AB = BA = E$$

Свойства:

① Ако A е обратима, то тогава $\exists! B: AB = BA = E$

Доказ:

~~Ако A и B изведнъж са~~

Ако B_1, B_2 изведнъж са:

$$A^{-1} = B_1 = B_1 E = B_1 (A B_2) = (B_1 A) B_2 = E B_2 = B_2$$

② A е обратима $\Rightarrow (A^{-1})^{-1} = A$

③ A - обратима, $c \neq 0$ ($c \in F$)

$$(cA)^{-1} = c^{-1}A^{-1}$$

④ Ако A, B са обратими, то AB е обратима, то

$$AB \neq (AB)^{-1} = A^{-1}B^{-1}$$

Th. 1 A е обратима $\Leftrightarrow \det A \neq 0$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

В стълба с номер k стоят адюнгираните
коэффициенти на редо с номер k .

Доказ-во:

\Rightarrow

$$A \text{ - обратима } \Rightarrow \exists B: AB = BA = E$$

$$\det AB = \det A \det B = \det E$$

$$\det A \cdot \det B = 1$$

$$\Rightarrow \det A \neq 0 \Rightarrow \det A^{-1} = \frac{1}{\det A}$$

\Leftarrow

$$\det A \neq 0 \Leftrightarrow \det A \neq 0$$

$$AB = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \frac{1}{\det A} \begin{pmatrix} A_{11} & \dots & A_{n1} \\ A_{12} & \dots & A_{n2} \\ \vdots & \vdots & \vdots \\ A_{1n} & \dots & A_{nn} \end{pmatrix} =$$

$$= \frac{1}{\det A} \begin{pmatrix} c_{11} & \dots & c_{1n} \\ - & - & - \\ c_{n1} & \dots & c_{nn} \end{pmatrix}$$

$$c_1 = a_{11} A_{11} + a_{12} A_{12} + \dots + a_{1n} A_{1n} =$$

$$\Rightarrow = \det A$$

$$c_2 = a_{11} A_{21} + a_{12} A_{22} + \dots + a_{1n} A_{2n} = 0$$

$$c_{ij} = a_{i1} A_{j1} + a_{i2} A_{j2} + \dots + a_{in} A_{jn} = \begin{cases} \det A, & i=j \\ 0, & i \neq j \end{cases}$$

$$\Rightarrow \frac{1}{\det A} \begin{pmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \ddots \\ & & \det A \end{pmatrix} \Rightarrow A \cdot B = E$$

$$BA = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{pmatrix} =$$

$$= \frac{1}{\det A} \begin{pmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & & 0 \\ & & \ddots & \\ 0 & \dots & & \det A \end{pmatrix} = E$$

~~то~~

$$\Rightarrow AB = BA = E \Rightarrow B = A^{-1}$$

\Rightarrow матрица A е обратима

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \det A = -2$$

$$A_{11} = 4; \quad A_{12} = -3; \quad A_{21} = -2; \quad A_{22} = 1$$

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

Th. Елементарните преобразувания по редове на една матрица могат да се реализират чрез умножаване отляво по подходяща неособена матрица. Елементарните преобразувания се реализират чрез умножение отдясно по неопределена матрица.

неособена матрица $n \times n$ е такава \Leftrightarrow $\left| \begin{array}{l} n \times n \\ \det \neq 0 \\ \text{не } 0 \\ \text{е обра-} \end{array} \right.$
матрицата е n ,
матрицата $\neq 0$
типа

~~Елементарните~~

~~$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$~~

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{B_1} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{i1} & \dots & \lambda a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{i1} & \dots & \lambda a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$

$$\det B_1 = \lambda \neq 0$$

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 0 & 0 \\ i & & 0 & 0 & 1 & \\ j & & 0 & 1 & & 0 \\ & & 0 & & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{in} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} \xrightarrow{\text{swap } i, j} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{ji} & \dots & a_{jn} \\ a_{ij} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$

B_2

$$\det B_2 = -1$$

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & 1 & -1 & 0 \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{in} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} \xrightarrow{R_i \leftarrow R_i + R_j} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} + a_{j1} & \dots & a_{in} + a_{jn} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$

B_3

$$\det B_3 = 2^{1,2,3}$$

Th. Нека $A \in M_{n \times n}$ и $\det A \neq 0$ ($r(A) = n$). С последователни елементарни преобразувания по редове A може да се приведе

го E , и ако тези преобр. в стъпка ред се умно-
жат, към E се получава A^{-1} .

$$B_t(\dots (B_2(B_1 A))) = E$$

$$\underbrace{(B_t \dots B_2 B_1)}_B A = E \Rightarrow BA = E \quad | \cdot A^{-1}$$

$$BA \cdot A^{-1} = E A^{-1}$$

$$\Rightarrow B = A^{-1}$$

$$B = A^{-1} = B_t / B_{t-1} (\dots (B_2 (B_1 E)))$$

$$(A | E) \xrightarrow[\text{по ред.}]{\text{преобр.}} (E | A^{-1})$$

$$\left(\begin{array}{c} A \\ E \end{array} \right) \xrightarrow[\text{по стълб}]{\text{преобр.}} \left(\begin{array}{c} E \\ A^{-1} \end{array} \right)$$

$$(A | C) \xrightarrow[\text{на редове}]{\text{преобр.}} (E | X = ?)$$

$$\left(\begin{array}{c} A \\ C \end{array} \right) \xrightarrow[\text{на стълб}]{\text{преобр.}} \left(\begin{array}{c} E \\ X = ? \end{array} \right)$$

14. Смяна на базис

V - крайномерно линейно пр-во с размерност
 n - $\dim V = n$